MATHEMATICS OF OPERATIONS RESEARCH
Vol. 19, No. 2, May 1994
Printed in U.S.A.

ON THE COMPLEXITY OF COOPERATIVE SOLUTION CONCEPTS

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We study from a complexity theoretic standpoint the various solution concepts arising in cooperative game theory. We use as a vehicle for this study a game in which the players are nodes of a graph with weights on the edges, and the value of a coalition is determined by the total weight of the edges contained in it. The Shapley value is always easy to compute. The core is easy to characterize when the game is convex, and is intractable (NP-complete) otherwise. Similar results are shown for the kernel, the nucleolus, the ϵ-core, and the bargaining set. As for the von Neumann-Morgenstern solution, we point out that its existence may not even be decidable. Many of these results generalize to the case in which the game is presented by a hypergraph with edges of size $k > 2$.

1. Introduction. Formalizing fairness in a cooperative environment is one of the fundamental conceptual problems in game and economic theory. To understand the issue, suppose that we have a finite set $N = \{1, 2, \dots, n\}$ of players. The players may form arbitrary *coalitions*, and for each possible coalition $S \subseteq N$ we know the amount $v(S)$ that coalition S cannot be prevented from obtaining ($v(S)$ is called the value of S). The basic problem is this: Given a proposed imputation $x = (x_1, \dots, x_i, \dots, x_n)$, where $\sum_{i=1}^{n} = v(N)$, is x a "fair" way for the n players to split $v(N)$? There are many such notions of "fairness" that have been proposed in the past; such notions are usually called *solution concepts* (the survey in Shubik (1981) lists at least ten). These solution concepts differ substantially in their naturalness, intuitiveness, sophistication, and the apparent complexity of their definition. There are arguments and counterarguments why each such proposal is a reasonable mathematical rendering of the intuitive concept of "fairness."

A solution concept defines, for each function v: $2^N \mapsto R$ (where it is assumed that $v(\emptyset) = 0$), a class $\mathscr{F}$ of imputations (that is, vectors in R^n summing to $v(N)$). Intuitively, an imputation is considered "fair" if it belongs in this class. There are at least three natural *computational* problems associated with a solution concept. Perhaps the most natural problem is to decide whether a proposed allocation is "fair" according to $\mathscr{F}$: (1) "Given an imputation x, does it belong to $\mathscr{F}$?" One may wish to ask whether there are any fair allocations at all, given the present coalition situation: (2) "Is $\mathscr{F}$ nonempty?" Problem (2) is interesting because it is related to the problem of generating *some* member of $\mathscr{F}$. In the case of the von Neumann-Morgenstern solution, one even has to ask (3) "Does $\mathscr{F}$ exist?" (as this is not always the case, see Lucas (1971)).

We propose to study the computational *complexity* (Papadimitriou (1993)) of the problems associated with each solution concept, and classify the concept as "simple" or "complex" depending on the outcome. That is, we propose another criterion for

Received April 4, 1989; revised November 3, 1992.
AMS 1980 subject classification. Primary: 90D12. Secondary: 68X25, 90C60.
IAOR 1973 subject classification.
OR/MS Index 1978 subject classification. Primary: 234 Games/group decisions/Cooperative. Secondary: 012 Analysis of algorithms/Computational complexity.
Key words. Cooperative game theory, computational complexity.

0364-765X/94/1902/0257/$01.25

judging whether a proposed solution concept is appropriate: *The computational complexity of the problems associated with it should not be too great*. There is something obviously unfair about a concept of "fairness" that requires a supercomputer in order to test whether it applies in a given situation, or in order to produce an example of an allocation that is fair according to the concept. But more importantly, our proposed criterion can be seen as an instance of the thesis of *bounded rationality* (see Simon (1972) for an extensive discussion). Bounded rationality is the hypothesis that decisions by realistic economic agents cannot involve unbounded resources for reasoning. It has been recently suggested (see, e.g., Kalai and Stanford (1988), Neyman (1985), Papadimitriou ( ), and also Futia (1977) for an earlier attempt) that computational complexity is an appropriate mathematical vehicle for capturing bounded rationality. Since solution concepts are proposed as the basis for economic decisions (deciding whether a proposed split is fair, or generating a fair split), the complexity of problems (1), (2) and (3) are of great interest in this regard.

Thus, we are led to the study of the computational problems (1), (2), and (3) above for given functions v. There is a catch, however: If the game is defined by the 2^n coalition values, there may be little to be said about the computational complexity of the various solution concepts, because the input is already exponential in n, and thus, in most cases, the computational problems above can be solved very "efficiently." In order to arrive at computationally meaningful questions, we focus on the following interesting case of the problem: We are given an *undirected graph* $G = (N, E)$, with an integer weight $v(i, j)$ on each edge $\{i, j\}$. We then define a game in which, for each coalition S, $v(S)$ is defined as $\sum_{\{i, j\} \subseteq S} v(i, j)$. That is, a coalition of nodes can guarantee for its members the weight of the subgraph of G induced by the coalition. We denote this game defined by the weighted graph G as v_G. This situation can be thought of as the problem of dividing fairly between n cities the income from a highway network connecting them.

Notice that, if all weights are nonnegative, this set function is subadditive, and furthermore the game is *convex* (Shubik 1981). In the next section we study the complexity of computational problems (1), (2), and (3) for various solution concepts in the game on the graph as defined above. Our results are roughly these: The Shapley value of the graph game is always easy to compute, as it reduces to half the sum of the weights of the edges adjacent to each node. For several other solution concepts we show that Problems (1) and (2) are NP-hard in general, although in some cases they can be reformulated as max-flow problems and solved efficiently when all weights are positive. For the bargaining set solution, problem (1) may not even be in NP (we conjecture that it is Π_2^p-complete). Finally, problem (3) for the von Neumann-Morgenstern solution is not even known to be decidable (we point out that the fragment of logic in which it is defined is indeed undecidable). In §3 we discuss briefly the more general situation, in which the values are given for all *unordered k-tuples* of nodes for integers $k \geqslant 2$ (that is, instead of a graph we have a *hypergraph* with edges of cardinality k).

2. Solution concepts. If $x \in R^n$ and $S \subseteq N$, we let $x(S) = \sum_{i \in S} x_i$, and $x(i) = x(\{i\}) = x_i$. Also let $v(S, T) = \sum_{i \in S,\, j \in T} v(i, j)$ and $e(S, x) = v(S) - x(S)$.

2.1. *The Shapley value*. The Shapley value of a game (Shapley 1972) is an imputation intended to reflect the marginal contribution of each player to the outcome, averaged over all possible "orders of arrival" of the players. That is, the proposed set $\mathscr{F}$ of fair solutions is a singleton, denoted ϕ, and defined as follows: For each player i, For each player i,

$$\phi(i) = \sum_{i \in S \subseteq N} \frac{(n - |S|)!(|S| - 1)!}{n!} (v(S) - v(S - \{i\})).$$

Since this definition involves all subsets of N which contain i, it is not easy to compute the Shapley value of a game in general. In the special case considered in this paper, however, an easy calculation shows that the Shapley value is easily computable:

THEOREM 1. *The Shapley value of game v_G is $\phi(i) = \frac{1}{2}\sum_{j \neq i} v(i, j)$.*

PROOF. Consider the contribution of edge $(i, j), v(i, j)$, to $\phi(i)$. For every subset S containing i, j, this edge contributes

$$\frac{(n - |S|)!(|S| - 1)!}{n!} v(i, j).$$

There are $\binom{n-2}{k-2}$ subsets S of size k that contain both i and j. All the subsets S of size k containing i, j contribute in total

$$\binom{n-1}{k-2} \frac{(n-k)!(k-1)!}{n!} v(i, j),$$

which is

$$\frac{k-1}{n(n-1)} v(i, j).$$

Summing over $k = 2, 3, \ldots, n$, we have $\phi(i) = \frac{1}{2}\sum_{j \neq i} v(i, j)$. □

Notice that, according to this result, the Shapley value is very easy to compute with $O(n^2)$ operations. Also, it is a very natural notion of fairness, since it assigns to each node half the weight (income) from each of its adjacent edges.

2.2. *The core.* The *core* of a game is the set of all imputations x such that, for all coalitions $S \subseteq N$, $v(S) \leqslant x(S)$.

We call the function $e(S, x) = v(S) - x(S)$, defined above, the excess of a coalition S at an imputation x. Thus, an imputation is in the core if and only if for any coalition S the excess is nonpositive. However, it is easy to see that the excess of S at the Shapley value, $e(S, \phi)$, is $-\frac{1}{2}$ times the total weight of the edges joining vertices in S with vertices in $N - S$; it is thus half the weight of the *cut* $(S, N - S)$. Hence we have:

LEMMA 1. *The Shapley value is in the core of v_G if and only if there is no negative cut in G.* □

In fact, we can show that the Shapley value is the most likely member of the core:

LEMMA 2. *The core of v_G is nonempty if and only if there is no negative cut in G.*

PROOF. The *if* part follows from Lemma 1.

Conversely, suppose we have a negative cut, i.e., for some subset S of N, $v(S, N - S) = \sum_{i \in S,\ j \in N-S} v(i, j) < 0$. Thus, $\phi(S) - v(S) = \phi(N - S) - v(N - S) = v(S, N - S)/2 < 0$. For any imputation x, we have

$$x(N) = x(S) + x(N - S) = v(N) = \phi(N) = \phi(S) + \phi(N - S).$$

So,

$$x(S) - v(S) + x(N - S) - v(N - S)$$
$$= \phi(S) - v(S) + \phi(N - S) - v(N - S)$$
$$= v(S, N - S) < 0.$$

Therefore, either $x(S) - v(S) < 0$ or $x(N - S) - v(N - S) < 0$. Consequently, x cannot be in the core, and the core is empty. □

To understand the computational complexity of Problems (1) and (2) for the core, we need the following easy complexity result, to our knowledge not observed before:

LEMMA 3. *It is* NP-*complete to tell whether a graph has a negative cut.*

PROOF. It is easy to see the problem is in NP. To prove it is NP-hard, we shall reduce the following known NP-complete problem to the negative cut problem:

> MAX-CUT: Given a graph $G = (V, E)$ with a nonnegative weight $c(i, j)$ on every edge (i, j) and given an integer $K > 0$, is there a cut in the graph with total weight $> K$?

Suppose that we are given such a graph G, a weight function c on its edge set, and an integer K. Let $c(E) = \sum_{\{i,j\} \in E} c(i, j)$, and let $c'(i, j) = -c(i, j)$. We add two new nodes 0 and $n + 1$ to the graph, with $c'(k, i) = c(E)$ for $k \in \{0, n + 1\}$ and $i = 1, 2, \ldots, n$. Finally, let $c'(0, n + 1) = K - nc(E)$. On the new graph G' with weight function c', any cut which does not separate 0 from $n + 1$ will have a nonnegative weight because there will be at least one edge of weight $c(E)$ in the cut, and the total sum of negative edges in the cut is no less than $-c(E)$. Hence, a negative cut will always separate 0 from $n + 1$ and will always cut exactly n edges of weight $c(E)$. Therefore, the remaining edges of the alleged negative cut induce a cut in the original graph of total weight $> K$. It follows that there is a negative cut in graph G' if and only if there was a cut of weight at least K in the original graph. □

THEOREM 2. *The following problems are* NP-*complete*:
(1) *Given v_G and imputation x, is it not in the core of v_G?*
(2) *Given v_G, is the Shapley value of v_G not in the core of v_G?*
(3) *Given v_G, is the core of v_G empty?*

PROOF. From Lemmas 1, 2, and 3, we know the above problems are NP-hard. To show that they are in NP, we notice if x is not in the core of v_G, then there is a subset S such that $x(S) < v_G(S)$, which can be checked in polynomial time. □

In contrast, if there are no negative edges in the graph (that is, if the game is convex), we can show the following fact:

LEMMA 4. *When all weights of G are nonnegative, we can test in polynomial time whether an imputation x is in the core of v_G.*

PROOF. We reduce the problem to a network flow problem. Given $G = (N, E)$ and weight v on E, we construct a flow network G'. The set of nodes of G' is $N' = N \cup E \cup \{0, n + 1\}$, where 0 is the source and $n + 1$ is the terminal. The directed arcs in the new graph are defined as follows: For each edge $\{i, j\}$ in E we add to G' arcs $(\{i, j\}, i)$, $(\{i, j\}, j)$, and $(0, \{i, j\})$ with capacities $c(\{i, j\}, i) = c(\{i, j\}, j) = \infty$ and $c(0, \{i, j\}) = v(i, j)$. For each node $i \in N$, we construct an arc $(i, n + 1)$ with weight $c(i, n + 1) = x(i)$.

We shall show that the value of the maximum flow in G' from 0 to $n + 1$ is $v(N) = x(N)$ if and only if x is in the core. First, suppose that there is a flow of this value. All arcs to $n + 1$ and all arcs from 0 are filled to capacity. Thus for any subset S of N there is flow through the corresponding nodes of G of value at least $x(S)$. However, this must come through the corresponding edges, and thus $x(S) \geqslant v(S)$; x is in the core. Conversely, suppose that we have a cut with capacity $c(S', N' - S') < v(N)$. Obviously, the cut cannot contain any of the arcs of capacity ∞. Let $S = N \cap S'$. Then $c(S', N' - S') = x(S) + v(N) - v(S) < v(N)$. Therefore $x(S) < v(S)$, and x is not in the core. □

THEOREM 3. *Problems* (1), (2), *and* (3) *in Theorem* 2 *can be solved in polynomial time if G has no negative edges.* □

The ϵ-core is another solution concept, which is in fact a relaxation of the core. The excess is not required to be nonpositive, but smaller than a number $\epsilon > 0$. Thus, the ϵ-core consists of all the imputations x such that $x(S) \geqslant v(S) - \epsilon$ for all $S \subseteq N$. Since we show in Theorem 6 below that the Shapley value of G minimizes the maximum excess, it follows that, for any number ϵ, the Shapley value is in the ϵ-core of v_G if and only if the ϵ-core of v_G is nonempty. Since telling whether there is a cut of weight larger than K is an NP-complete problem, we have:

THEOREM 4. *It is* NP-*complete, given G and* $\epsilon > 0$, *to tell whether any given imputation (and the Shapley value in particular) is not in the* ϵ-*core of* v_G. *It is also* NP-*complete, given G, to compute the smallest* ϵ *such that the* ϵ-*core is nonempty.* □

The same result holds for the *weak* ϵ-core, the set of all imputations x for which $x(S) \geqslant v(S) - \epsilon|S|$ for all $S \subseteq N$.

2.3. *The kernel and the nucleolus.* The kernel of a game consists of all imputations x such that for any two players i, j,

$$\max_{i \in S,\, j \notin S} e(S, x) = \max_{j \in S,\, i \notin S} e(S, x).$$

The kernel of v_G is a concept with a good computational characterization: The emptyness problem for the kernel of v_G is trivial, and the membership problem is also easy when the values are all nonnegative.

THEOREM 5. *The Shapley value is always in the kernel of* v_G.

PROOF. Since the nucleolus of a cooperative game is always in the kernel (Szep and Forgo 1985, Wang 1988), this result follows from Theorem 6 below. □

In contrast, we conjecture that telling whether an arbitrary imputation for general graphs is in the kernel is NP-hard.

Fix an imputation x, and define the quantity $e_1(S) = v(S) - x(S)$. Now order the subsets of N according to e_1: $e_1(S_1) \geqslant e_1(S_2) \geqslant \cdots \geqslant e_1(S_m)$, where $m = 2^n$. The nucleolus is the imputation which lexicographically minimizes the vector $(e_1(S_1), e_1(S_2), \dots, e_1(S_m))$. The complexity of computing the nucleolus (as it turns out, there is only one nucleolus, see Shubik 1981) is an open problem for general games. However, it is simplified considerably in the special case of graphs:

THEOREM 6. *The Shapley value of* v_G *is the same as the nucleolus.*

PROOF. Let x be the nucleolus; define $e_2(S) = v(S) - \phi(S)$, and suppose that we order the subsets of N: $e_2(T_1) \geqslant e_2(T_2) \geqslant \cdots \geqslant e_2(T_m)$. Let $\alpha_1 = (e_1(S_1), e_1(S_2), \dots, e_1(S_m))$, and $\alpha_2 = (e_2(T_1), e_2(T_2), \dots, e_2(T_m))$. Since x is the nucleolus, α_1 is lexicographically less than or equal to α_2. Since the excess of the Shapley value of a coalition equals the total weight of all outgoing edges, we have $e_2(S) = e_2(N - S)$. Therefore, we can arrange the ordering of the T_j's such that $T_{2i-1} = N - T_{2i}$. We shall then have $e_2(T_{2i-1}) = e_2(T_{2i})$, for each $i = 1, 2, \dots, m/2$; we can assume without loss of generality that $e_1(T_{2i-1}) \geqslant e_1(T_{2i})$.

We claim that for each $i = 1, 2, \dots, m/2$,

$$e_1(S_{2i-1}) = e_1(S_{2i}) = e_1(T_{2i-1}) = e_1(T_{2i}) = e_2(T_{2i-1}) = e_2(T_{2i}). \tag{1}$$

The proof is by imputation on i. First, when $i = 1$, we claim that $e_1(T_1) \leqslant e_1(S_1) \leqslant e_2(T_1) = e_2(T_2) \leqslant e_1(T_2) \leqslant e_1(S_2) \leqslant e_1(S_1)$. The first inequality is from the fact that the S's are sorted in decreasing order; the second from the fact that x is the nucleolus; the third equality is by the symmetry of the excess of the Shapley value; the fourth inequality from the observation that $e_1(T_1) + e_1(T_2) = e_2(T_1) + e_2(T_2) = v(T_1) + v(T_2) - v(N)$ and the previous inequalities; the fifth inequality from the fact that the S's are sorted in decreasing order and the fact that $e_1(T_1) \geqslant e_1(T_2)$; and the last from the fact that the S's are sorted in decreasing order. Comparing terms, we notice that all except for the first must be equal, and the first is equal to them because of $e_1(T_1) + e_1(T_2) = e_2(T_1) + e_2(T_2)$.

Suppose, for all $i < k$, (1) holds. The induction step is identical to the previous argument. Because x is in the nucleolus and by the inductive hypothesis that the first $2k - 2$ elements of α_1 and α_2 are the same, we have $e_1(S_{2k-1}) \leqslant e_2(T_{2k-1})$. By the induction hypothesis, $e_1(T_i) = e_1(S_i)$, $i = 1, 2, \dots, 2k - 2$. From the ordering of vector α_1, we have $e_1(T_{2k-j}) \leqslant e_1(S_{2k-1})$, $j = 0, 1$. Since $e_2(T_{2k}) = e_2(T_{2k-1})$, $e_1(T_{2k-j}) \leqslant e_1(S_{2k-1}) \leqslant e_2(T_{2k-j})$, $j = 0, 1$. Again by summing over both sides of the two inequalities, both sides are equal to $v(T_{2k-1}) + v(T_{2k}) - v(N)$, which requires that equalities hold in the original two inequalities. Thus, $e_1(T_{2k-j}) = e_1(S_{2k-1}) = e_2(T_{2k-j})$, $j = 0, 1$. Since $e_1(S_{2k-1}) \geqslant e_1(S_{2k}) \geqslant \min[e_1(T_{2k-1}), e_1(T_{2k-2})]$, (1) holds for $i = k$.

Therefore, (1) is true for all $i = 1, 2, \dots, m/2$. In other words, $e_1(T) = e_2(T)$ for every subset T of N. In particular, let $T = \{j\}$ for $j \in N$, and $v(j) - x(j) = v(j) - \phi(j)$. Thus $x(j) = \phi(j)$ for every $j \in N$. □

2.4. *The bargaining set.* The *bargaining set* is the set of all imputations x for which, for all players i and j, if there is an imputation y and a coalition S containing i but not j, and such that (a) $y(S) \leqslant v(S)$, and (b) $x_k < y_k$ for all $k \in S$, then there is another imputation z and set T containing j but not i such that (a) $z(T) \leqslant v(T)$; (b) $z_k \geqslant y_k$ for $k \in T \cap S$, and (c) $z_k \geqslant x_k$ for $k \in T - S$. Intuitively, this means that whenever a player i objects to x using the arguments y and S, then j can respond with arguments z and T.

THEOREM 7. *Deciding whether imputation is in the bargaining set of v_G is* NP-*hard*.

PROOF. We prove that the problem is NP-hard, again, by a reduction from the negative cut problem to the bargaining set problem. Suppose we are given a graph $G = (V, E)$ and a weight function v: $E \mapsto R$. Recall that the Shapley value is not in the core iff there is a negative cut.

Consider the Shapley value of v_G, and construct a new graph H by adding a new node u^0 to G and assigning weight $w(|E|)$ to each of the new edges between u^0 and all the old nodes, where $w(|E|) = \sum_{e \in E} |v(e)|$. Define an imputation x of v_H by assigning the Shapley value of v_G to the nodes of G, and assigning the sum of the edge weights of all the new edges to u^0. We denote this imputation by x_0. Since every cut C should contain at least one of the edges incident to u^0 which has a weight

$w(|E|)$, and the total weight of all the other edges in the cut C is at least $-w(|E|)$, it follows that H has no negative cut.

We claim that x_0 is in the core of v_H iff G has no negative cut. For the if-part, suppose G has no negative cut, and let S be a subset of vertices of H. If $u^0 \notin S$, then $x_0(S) = \phi_G(S)$ and $v_H(S) = v_G(S)$. Since ϕ_G is in the core of G when G has no negative cut, we see that $x_0(S) = \phi_G(S) \geqslant v_G(S) = v_H(S)$. If $u^0 \in S$, then $x_0(S - \{u^0\}) \geqslant v_H(S - \{u^0\})$ from the above discussion. Moreover, by the assignment of $x_0(u^0)$, $x_0(u^0) = |V| \cdot w(|E|) \geqslant |S - \{u^0\}| \cdot w(|E|) = v_H(S - \{u^0\}, u^0)$. Adding these two inequalities, we again obtain $x_0(S) \geqslant v_H(S)$. Thus, x_0 is in the core of H. For the only-if-part, assume that G has a negative cut. Then, there must be a subset $S \subset V$ such that $\phi_G(S) < v_G(S)$ since the core of G is empty in this case. Thus, $x_0(S) = \phi_G(S) < v_G(S) = v_H(S)$ and x_0 is not in the core of H.

From $H = (V', E')$ and the imputation x_0 on v_H, we further construct a new graph D as follows. We add a new node u^1, and new edges $\{(i, u^1)\colon i \in V'\}$. The edge weight for the new edges is defined to be $\phi_H(i) - x_0(i)$ for each edge, (i, u^1), where $\phi_H(i)$ is the Shapley value on the graph H. Define a new imputation y_0 by assigning the Shapley value of H to the nodes in H, and 0 to the new node u^1. We claim that y_0 is in the bargaining set of D, iff x_0 is in the core of H.

If x_0 is not in the core of H, then, G has a negative cut. Thus, there is a subset $S \subset V$ such that, $\phi_G(S) < v_G(S)$. Consider the subset $S' = S \cup \{u^1\}$ in the graph D now: $y_0(S') = \phi_H(S)$ and $v_D(S') = \phi_H(S) - x_0(S) + v_H(S)$. Thus, $y_0(S') - v_D(S') = x_0(S) - v_H(S)$. Since $S \subset V$, $x_0(S) = \phi_G(S)$ and $v_H(S) = v_G(S)$. We have $y_0(S') - v_D(S') = x_0(S) - v_H(S) = \phi_G(S) - v_G(S) < 0$. However, for any subset $T \subset V'$, $y_0(T) = \phi_H(T)$ and $v_D(T) = v_H(T)$. Since there is no negative subset in H, $y_0(T) - v_D(T) = \phi_H(T) - v_H(T) = \mathrm{cut}(H, \overline{H}) \geqslant 0$. Thus, in the above definition for the bargaining set, take $i = u^1$ and $j \notin S$, then i can object to y_0 with the subset S' and any imputation y_1 with $y_1(i) = y_0(i) + \epsilon$, $\forall i \in S'$, for a sufficiently small $\epsilon > 0$. The player j cannot respond with a subset T which does not contain u^1. Thus, y_0 is not in the bargaining set. On the other hand, if x_0 is in the core of H, then G has no negative cuts. We have $y_0(S') - v_D(S') = x_0(S) - v_H(S) = \phi_G(S) - v_G(S) \geqslant 0$ for all subset S' with $u^1 \in S'$. And also as we have seen, for all T with $u^1 \notin T$, $y_0(T) - v_D(T) = \phi_H(T) - v_H(T) = \mathrm{cut}(H, \overline{H}) \geqslant 0$. The imputation y_0 is in the core of D and thus in the bargaining set of D.

Combining the two claims proven above, we conclude that y_0 is not in the bargaining set of D iff G has a negative cut. Therefore, it is NP-hard to decide if an imputation is in the bargaining set. □

We conjecture that the problem of deciding whether an imputation is in the bargaining set of v_G is Π_2^p-complete. The complexity class Π_2^p, a superset of NP, contains sets expressible as $\{x\colon \forall y \; \exists z R(x, y, z)\}$, where R is a polynomially computable and polynomially balanced (that is, the length of y and z is polynomial in the length of x) ternary predicate (see Papadimitriou (1993) for definitions and a detailed treatment of the subject). Telling whether an imputation is in the bargaining set of v_G is in Π_2^p, since the definition of the bargaining set has this logical structure of two alternating quantifiers, universal first.

2.5. *On the von Neumann-Morgenstern solution.* In their classical work in Game Theory, von Neumann and Morgenstern defined the (historically first) solution concept. Suppose that x and y are imputations. We say that x *dominates* y if there is a coalition S such that (a) $x(S) \leqslant v(S)$, and (b) $x_i > y_i$, $i \in S$. We say that a set $\mathscr{F}$ of imputations is a *solution* if (1) no two imputations in $\mathscr{F}$ dominate each other, and (2) any imputation not in $\mathscr{F}$ is dominated by some imputation in $\mathscr{F}$. This solution concept lost its early popularity when it was shown (Lucas) that there are games with no solutions.

When the game is convex (in our case, when the edge weights are positive), then the core is a solution. In general, however, one has to rely on the definition of the von Neumann-Morgenstern solution in order to determine whether a game v_G has one. It is not obvious at all that there is *any* algorithm (however slow) for deciding this!

Logically, the existence of a van Neumann-Morgenstern solution can be expressed as follows:

$$\exists \mathscr{F} \subset \mathfrak{R}^n \ \forall \vec{x} \mathscr{L}(x, \mathscr{F}, v),$$

where $\mathscr{L}(x, \mathscr{F}, v)$ is a Boolean combination of linear inequalities involving the variables x_i, and the constants v_i, as well as statements of the form $(x_1, x_3, v_2) \in \mathscr{F}$. It is worth remarking that all other solution concepts described above define $\mathscr{F}$ in terms of sentences of considerably simpler structure (most significantly, without the second order existential quantifier). In fact, it may be interesting to note that the problem of checking the validity of sentences such as the one above is *undecidable* (even in the $n = 3$ case). The proof (due to Sam Buss) involves using the linear inequalities in $\mathscr{L}$ for encoding in the set $\mathscr{F}$ of vectors the multiplication table of the integers (by stating in $\mathscr{L}$ the axioms that define multiplication), thus reducing the validity problem for Number Theory, shown undecidable in Gödel's (1931) classical work, to the problem in hand.

3. Extensions. An interesting generalization of the situation considered is the one in which $v(S)$ is given as the sum of $v(e)$ not over all sets e of size 2 contained in S, but over all sets of size $k \geqslant 2$ contained in S; notice that v is still presented by data that are polynomial in n, and hence the complexity questions are still meaningful. In this case we can think that v is given in terms of a weighted *hypergraph* $H = (N, E)$, with hyperedges of size k (or, we may even let the size of a hyperedge vary, as long as the number of hyperedges is polynomial in n; we do not elaborate more on this generalization). It turns out that in this generalization the Shapley value can again be computed easily as $\phi(i) = \frac{1}{k}\sum_{i \in e \in E} v(e)$. The excess of the Shapley value at a set S is $\frac{i}{k}$ times the total weight of the hyperedges having exactly i nodes in S, summed over $i = 1, \ldots, k$. It is not hard to extend the proof of Theorem 2 to show that telling whether the Shapley value is in the core (that is, whether in a hypergraph there is a set S with a negative such sum) is NP-complete.

On the positive side, when all weights are nonnegative the MAX-FLOW technique of Lemma 4 still applies: We shall show how to decide in polynomial time whether a given imputation x is in the core. For the flow network H', the set of nodes is $N' = N \cup E \cup \{0, n + 1\}$, where 0 is the source and $n + 1$ is the terminal. The directed arcs in the new graph are defined as follows: For each edge $e \in E$ we add to H' arcs (e, i) for all $i \in e$, and $(0, e)$, with capacities $c(e, i) = \infty$ and $c(0, e) = v(e)$. For each node $i \in N$, we construct an arc $(i, n + 1)$ with weight $c(i, n + 1) = x(i)$. Thus:

THEOREM 8. *Problems* (1), (2), *and* (3) *in Theorem* 2 *are* NP-*complete in the case of hypergraphs, and can be solved in polynomial time for hypergraphs with no negative edges.* □

A natural question is whether Theorem 5 and Theorem 6 (relating the Shapley value to the kernel and the nucleolus) still hold for hypergraphs. Theorem 5 does not,

not even in the convex case. First notice that

$$v(S) - \phi(S) = - \sum_{e \in (S, N-S)} \frac{|e \cap S|}{|e|} v(e),$$

where $(S, N-S)$ is the set of all hyperedges that intersect both S and $N-S$. Consider a game with four players, $\{1,2,3,4\}$, with $k = 3$ and $v(1,2,3) = 3$; and $v(S) = 1$, if $|S| = 3$ and $4 \in S$. Thus $v(1,2,3,4) = 6$. One can easily calculate that $\max_{1 \in S, 4 \notin S} e(S, \phi) = -\frac{4}{3}$ while $\max_{4 \in S, 1 \notin S} e(S, \phi) = -1$. Hence the Shapley value is not in the kernel. Since the nucleolus is always in the kernel, it also follows that the Shapley value is not the nucleolus of this game.

Finally, as with all complexity results, ours depend on the chosen representation of the input (that is, the game). They are informative results about the complexity of solution concepts only to the extent that games such as v_G can be considered representative instances of the more general framework. A more general representation would be the one in which the game is given implicitly by an algorithm for computing $v(S)$, given any S. It is clear that our NP-hardness results (Theorems 2, 4, and 7) also hold in this more general situation (since the graph game studied can be seen as a special case of this). Our positive results (Theorems 1, 3, 5, and 6), however, may or may not hold in this more general setting. In other special cases and representations, completely different results are possible.

As an illustration of this point, let us consider the following game, of a completely different nature: *Weighted majority games*. Each player $i \in N$ has an assigned positive integer weight w_i, and we let $W = \sum_{i=1}^{n} w_i$. The value of a coalition S is one if $\sum_{i \in S} w_i > \frac{W}{2}$, and zero otherwise. In this game problems (1), (2), and (3) in Theorem 2 are all trivial (the core is empty if there are two distinct minimal majorities, and all other cases are trivial). In contrast, computing the Shapley value is #P-complete, that is, as hard as any counting problem in NP, see Papadimitriou (1993).

THEOREM 9. *Computing the Shapley value in weighted majority games is* #P-*complete*.

PROOF. Computing $n!\phi(i)$ can be considered as the number of accepting computations of a nondeterministic Turing machine, so the problem is in the class #P. To show completeness, recall the problem KNAPSACK (Papadimitriou and Steiglitz 1982, Papadimitriou 1993): We are given positive integers $a_1, \ldots, a_m$, and another K, and we are asked whether there is a subset S of $\{1, 2, \ldots, m\}$ such that $\sum_{i \in S} a_i = K$. It is easy to see from the reductions explained in (Papadimitriou and Steiglitz 1982, Papadimitriou 1993) that this problem is NP-complete even if (1) $K = \frac{M}{2}$, where $M = \sum_{i=1}^{m} a_i$; and (2) for some $k < m$, if $\sum_{i \in S} a_i = K$ then $|S| = k$; that is, all solutions have the same cardinality. Now, *the counting problem* associated with such instances of KNAPSACK (the problem of determining how many solutions exist) is #P-complete. We shall reduce this problem to computing the Shapley value of a weighted majority game.

Given such an instance of KNAPSACK with m integers $a_1, \ldots, a_m$, we construct the following majority game: $n = m + 1$; for $i = 1, \ldots, m$, $w_i = a_i$; and $w_n = 1$. It is now easy to see that for any subset S of $N v(S) - v(S - \{n\})$ is one if and only if $n \in S$,

$$\sum_{i \in S} w_j > \frac{M+1}{2} \quad \text{and} \quad \sum_{i \in S - \{n\}} w_j < \frac{M+1}{2}.$$

But, since $w_n = 1$, this is the same as saying that

$$\sum_{i \in S - \{n\}} w_j = \frac{M}{2} = K,$$

and thus $S - \{n\}$ is a solution to the original instance of KNAPSACK. $v(S) - v(S - \{n\})$ is zero otherwise. It follows from the definition of the Shapley value (see the beginning of §2) that $\phi(n)$ is precisely

$$\frac{(n-k)!(k-1)!}{n!}$$

times the number of solutions of the given instance of KNAPSACK (where k is the integer guaranteed by property (2) above). □

Acknowledgement. We are grateful to the editor and two anonymous referees for their very careful and justly critical reading of our manuscript, as well as for suggesting the current stronger form of Theorem 6, and the extensions in §3.

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