Settling the Complexity of Computing Two-Player Nash Equilibria

Xi Chen^{*}

Xiaotie Deng[†] Shang-Hua Teng[‡]

Abstract

We settle a long-standing open question in algorithmic game theory. We prove that BI-MATRIX, the problem of finding a Nash equilibrium in a two-player game, is complete for the complexity class **PPAD** (Polynomial Parity Argument, Directed version) introduced by Papadimitriou in 1991.

This is the first of a series of results concerning the complexity of Nash equilibria. In particular, we prove the following theorems:

- BIMATRIX does not have a fully polynomial-time approximation scheme unless every problem in **PPAD** is solvable in polynomial time.
- The smoothed complexity of the classic Lemke-Howson algorithm and, in fact, of any algorithm for BIMATRIX is not polynomial unless every problem in **PPAD** is solvable in randomized polynomial time.

Our results demonstrate that, even in the simplest form of non-cooperative games, equilibrium computation and approximation are polynomial-time equivalent to fixed point computation. Our results also have two broad complexity implications in mathematical economics and operations research:

- Arrow-Debreu market equilibria are **PPAD**-hard to compute.
- The P-Matrix Linear Complementary Problem is computationally harder than convex programming unless every problem in **PPAD** is solvable in polynomial time.

^{*}Department of Computer Science, Tsinghua University, Beijing, P.R.China. email: csxichen@gmail.com

[†]Department of Computer Science, City University of Hong Kong, Hong Kong SAR, P.R. China. email: deng@cs.cityu.edu.hk

[‡]Department of Computer Science, Boston University, Boston and Akamai Technologies Inc., Cambridge, MA, USA. email:steng@cs.bu.edu

1 Introduction

In 1944, Morgenstern and von Neumann [43] initiated the study of game theory and its applications to economic behavior. At the center of their study was von Neumann's minimax equilibrium solution for two-player zero-sum games [56]. In a two-player zero-sum game, one player's gain is equal to the loss of the other. They observed that any general *n*-player (non-zero-sum) game can be reduced to an (n + 1)-player zero-sum game. Their work went on to introduce the notion of cooperative games and the solution concept of stable sets.

In 1950, following the original spirit of Morgenstern and von Neumann's work on two-player zero-sum games, Nash [45, 44] formulated a solution concept for non-cooperative games among multiple players. In a non-cooperative game, the zero-sum condition is relaxed and no communication and coalition among players are allowed. Building on the notion of mixed strategies of [56], the solution concept, now commonly referred to as the *Nash equilibrium*, captures the notion of the individual rationality of players at an equilibrium point. In a Nash equilibrium, each player's strategy is a best response to other players' strategies. Nash proved that every n-player, finite, non-cooperative game has an equilibrium point. His original proof [45, 39] was based on Brouwer's Fixed Point Theorem [7]. David Gale suggested the use of Kakutani's Fixed Point Theorem [30] to simplify the proof. Mathematically, von Neumann's Minimax Theorem for two-player zero-sum games can be proved by linear programming duality. In contrast, the fixed point approach to Nash's Equilibrium Theorem seems to be necessary: even for two-player non-cooperative games, linear programming duality is no longer applicable.

Nash's equilibrium concept has had a tremendous influence on economics, as well as in other social and natural science disciplines [27]. Nash's approach to non-cooperative games has played an essential role in shaping mathematical economics, which consider agents with competing individual interests. A few years after Nash's work, Arrow and Debreu [3], also applying fixed point theorems, proved a general existence theorem for market equilibria. Since then, various forms of equilibrium theorems have been established via fixed point theorems.

However, the existence proofs based on fixed point theorems do not usually lead to efficient algorithms for finding equilibria. In fact, in spite of many remarkable breakthroughs in algorithmic game theory and mathematical programming, answers to several fundamental questions about the computation of Nash and Arrow-Debreu equilibria remain elusive. The most notable open problem is that of deciding whether the problem of finding an equilibrium point in a two-player game is solvable in polynomial time.

In this paper, we settle the complexity of computing a two-player Nash equilibrium and answer two central questions regarding the approximation and smoothed complexity of this game theoretic problem. In the next few subsections, we will review previous work on the computation of Nash equilibria, state our main results, and discuss their extensions to the computation of market equilibria.

1.1 Finite-Step Equilibrium Algorithms

Since Nash and Arrow-Debreu's pioneering work, great progress has been made in the effort to find constructive and algorithmic proofs of equilibrium theorems. The advances for equilibrium computation can be chronologically classified according to the following two periods:

- **Finite-step period**: In this period, the main objective was to design equilibrium algorithms that terminate in a finite number of steps and to understand for which equilibrium problems finite-step algorithms do not exist.
- **Polynomial-time period**: In this period, the main objective has been to develop polynomialtime algorithms for computing equilibria and to characterize the complexity of equilibrium computation.

The duality-based proof of the minimax theorem leads to a linear programming formulation of the problem of finding an equilibrium in a two-player zero-sum game. One can apply the simplex algorithm, in a finite number of steps in the Turing model¹, to compute an equilibrium in a two-player zero-sum game with rational payoffs. A decade or so after Nash's seminal work, Lemke and Howson [38] developed a path-following, simplex-like algorithm for finding a Nash equilibrium in a general two-player game. Like the simplex algorithm, their algorithm terminates in a finite number of steps for a two-player game with rational payoffs.

The Lemke-Howson algorithm has been extended to non-cooperative games with more than two players [57]. However, due to Nash's observation that there are rational three-player games all of whose equilibria are irrational, finite-step algorithms become harder to obtain for games with three or more players. For those multi-player games, no finite-step algorithm exists in the classical Turing model.

Similarly, some exchange economies do not have any rational Arrow-Debreu equilibria. The absence of a rational equilibrium underscores the continuous nature of equilibrium computation. Brouwer's Fixed Point Theorem — that any continuous map f from a convex compact body, such as a simplex or a hypercube, to itself has a fixed point — is inherently continuous. Mathematically, the continuous nature does not hinder the definition of search problems for finding equilibria and fixed points. But to measure the computational complexity of these continuous problems in the classical Turing model, some imprecision or inaccuracy must be introduced to ensure the existence of a solution with a finite description [51, 52, 46, 26, 21]. For example, one possible definition of an approximate fixed point of a continuous map f is a point \mathbf{x} in the convex body such that $||f(\mathbf{x}) - \mathbf{x}|| \leq \epsilon$ for a given $\epsilon > 0$ [51].

In 1928, Sperner [53] discovered a discrete fixed point theorem that led to one of the most elegant proofs of the Brouwer's Fixed Point Theorem. Suppose that Ω is a *d*-dimensional simplex with vertices $v_1, v_2, ..., v_{d+1}$, and that S is a simplicial decomposition of Ω . Suppose Π assigns to each vertex of S a color from $\{1, 2, ..., d + 1\}$ such that, for every vertex v of S, $\Pi(v) \neq i$ if the i^{th} component of the barycentric coordinate of v, in terms of $v_1, v_2, ..., v_{d+1}$, is 0. Then,

¹The simplex algorithm also terminates in a finite number of steps in various computational models involving real numbers, such as the Blum-Shub-Smale model [5].

Sperner's Lemma asserts that there exists a simplex cell in S that contains all colors. This fully-colored simplex cell is often referred to as a *panchromatric simplex* or a *Sperner simplex* of (S, Π) . Consider a Brouwer map f with Lipschitz constant L over the simplex Ω . Suppose further that the diameter of each simplex cell in S is at most ϵ/L . Then, one can define a color assignment Π_f such that each panchromatric simplex in (S, Π_f) must have a vertex \mathbf{v} satisfying $||f(\mathbf{v}) - \mathbf{v}|| \leq \Theta(\epsilon)$. Thus, a panchromatic simplex of (S, Π_f) can be viewed as an approximate, discrete fixed point of f.

Inspired by the Lemke-Howson algorithm, Scarf developed a path-following algorithm, using simplicial subdivision, for computing approximate fixed points [51] and competitive equilibrium prices [52]. The path-following method has also had extensive applications to mathematical programming and has since grown into an algorithm-design paradigm in optimization and equilibrium analysis. One can take a similar approximation approach to study the complexity of Nash equilibria, especially for games involving three or more players.

1.2 Computational Complexity of Nash Equilibria

Since 1960s, the theory of computation has shifted its focus from whether problems can be solved on a computer to how efficiently problems can be solved on a computer. The field has gained maturity with rapid advances in algorithm design, algorithm analysis, and complexity theory. Problems are categorized into complexity classes, capturing the potential difficulty of decision, search, and optimization problems. The complexity classes \mathbf{P} , \mathbf{RP} , and \mathbf{BPP} , and their search counterparts such as \mathbf{FP} , have become the standard classes for characterizing computational problems that are tractable².

The desire to find fast and polynomial-time algorithms for computing equilibria has been greatly enhanced with the rise of the Internet [48]. The rise has created a surge of human activities that make computation, communication and optimization of participating agents accessible at microeconomic levels. Efficient computation is instrumental to support the basic operations, such as pricing, in this large scale on-line market [49]. Many new game and economic problems have been introduced, and in the meantime, classical game and economic problems have become the subjects for active complexity studies [48]. Algorithmic game theory has grown into a highly interdisciplinary field intersecting economics, mathematics, operations research, numerical analysis, and computer science.

In 1979, Khachiyan made a ground-breaking discovery that the ellipsoid algorithm can solve a linear program in polynomial time [34]. Shortly after, Karmarkar improved the complexity for solving linear programming with his path-following, interior-point algorithm [32]. His work initiated the implementation of theoretically-sound linear programming algorithms. Motivated by a grand challenge in Theory of Computing [16], Spielman and Teng [54] introduced a new algorithm analysis framework, *smoothed analysis*, based on perturbation theory, to provide rigorous complexity-theoretic justification for the good practical performance of the simplex algorithm.

 $^{{}^{2}\}mathbf{FP}$ stands for Function Polynomial-Time. In this paper, as we only consider search problems, without further notice, we will (ab)use **P** and **RP** to denote the classes of search problems that can be solved in polynomial time or in randomized polynomial-time, respectively. We believe doing so will help more general readers.

They proved that although almost all known simplex algorithms have exponential worst-case complexity [35], the smoothed complexity of the simplex algorithm with the shadow-vertex pivoting rule is polynomial. As a result of these developments in linear programming, equilibrium solutions of two-player zero-sum games can be found in polynomial time using the ellipsoid or interior-point algorithms and in smoothed polynomial time using the simplex algorithm.

However, no polynomial-time algorithm has been found for computing discrete fixed points or approximate fixed points, rendering the equilibrium proofs based on fixed point theorems non-constructive in the view of polynomial-time computability.

The difficulty of discrete fixed point computation is partially justified in the query model. In 1989, Hirsch, Papadimitriou, and Vavasis [26] proved an exponential lower bound on the number of function evaluations necessary to find a discrete fixed point, even in two dimensions, assuming algorithms only have a black-box access to the fixed point function. Their bound has recently been improved [8] and extended to the randomized query model [13] and to the quantum query model [23, 13].

Motivated by the pivoting structure used in the Lemke-Howson algorithm, Papadimitriou introduced the complexity class **PPAD** [46]. **PPAD** is an abbreviation for *Polynomial Parity Argument in a Directed graph*. He introduced several search problems concerning the computation of discrete fixed points. For example, he defined the problem SPERNER to be the search problem of finding a Sperner simplex given a polynomial-sized circuit for assigning colors to a particular simplicial decomposition of a hypercube. Extending the model of [26], he also defined a search problem for computing approximate Brouwer fixed points. He proved that even in three dimensions, these fixed point problems are complete for the **PPAD** class. Recently, Chen and Deng [9] proved that the problem of finding a discrete fixed point in two dimensions is also complete for **PPAD**.

In [46], Papadimitriou also proved that BIMATRIX, the problem of finding a Nash equilibrium in a two-player game with rational payoffs is member of **PPAD**. His proof can be extended to show that finding a (properly defined) approximate equilibrium in a non-cooperative game among three or more players is also in **PPAD**. Thus, if these problems are **PPAD**-complete, then the problem of finding an equilibrium is polynomial-time equivalent to the search problem for finding a discrete fixed point.

It is conceivable that Nash equilibria might be easier to compute than discrete fixed points. In fact, by taking advantage of the special structure of Nash's normal form games, Lipton, Markarkis, and Mehta [40] developed a sub-exponential time algorithm for finding an approximate Nash equilibrium. In their notion of an ϵ -approximate Nash equilibrium, for a positive parameter ϵ , each players' strategy is at most an additive ϵ worse than the best response to other players' strategies. They proved that if all payoffs are in [0, 1], then an ϵ -approximate Nash equilibrium can be found in $n^{O(\log n/\epsilon^2)}$ time.

In a complexity-theoretic breakthrough, Daskalakis, Goldberg and Papadimitriou [18] proved that the problem of computing a Nash equilibrium in a game among four or more players is complete for **PPAD**. To cope with the fact that equilibria may not be rational, they considered an approximation version of equilibria by allowing exponentially small errors. The complexity result was soon extended to the three-player game independently by Chen and Deng [10] and Daskalakis and Papadimitriou [20], with different proofs. The reduction of [18] has two steps: First, it reduces a **PPAD**-complete discrete fixed point problem, named 3-DIMENSIONAL BROUWER, to the problem of finding a Nash equilibrium in a degree-three graphical game [33]. Then, it reduces the graphical game to a four-player game, using a result of Goldberg and Papadimitriou [25]. This reduction cleverly encodes fixed points by Nash equilibria.

The results of [18, 10, 20] characterize the complexity of computing k-player Nash equilibria for $k \geq 3$. They also show that the fixed point approach is necessary in proving Nash's Equilibrium Theorem, at least for games among three or more players. However, these latest complexity advances on the three/four-player games have fallen short on the two-player game.

1.3 Computing Two-Player Nash Equilibria and Smoothed Complexity

There have been amazing parallels between discoveries concerning the two-player zero-sum game and the general two-player game. First, von Neumann proved the existence of an equilibrium for the zero-sum game, then Nash did the same for the general game. Both classes of games have rational equilibria when payoffs are rational. Second, more than a decade after von Neumann's Minimax Theorem, Dantzig developed the simplex algorithm, which can find a solution of a twoplayer zero-sum game in a finite number of steps. A decade or so after Nash's work, Lemke and Howson developed their finite-step algorithm for BIMATRIX. Then, about a quarter century after their respective developments, both the simplex algorithm [35] and the Lemke-Howson algorithm [50] were shown to have exponential worst-case complexity.

A half century after von Neumann's Minimax Theorem, Khachiyan proved that the ellipsoid algorithm can solve a linear program and hence can find a solution of a two-player zero-sum game with rational payoffs in polynomial time. Shortly after that, Borgwardt [6] showed that the simplex algorithm has polynomial average-case complexity. Then, Spielman and Teng [54] proved that the smoothed complexity of the simplex algorithm is polynomial. If history is of any guide, then a half century after Nash's Equilibrium Theorem, one should be quite optimistic to prove the following two natural conjectures.

- *Polynomial 2*-NASH *Conjecture*: There exists a (weakly) polynomial-time algorithm for BIMATRIX.
- *Smoothed Lemke-Howson Conjecture*: The smoothed complexity of the Lemke-Howson algorithm for BIMATRIX is polynomial.

An upbeat attitude toward the first conjecture has been encouraged by the following two facts. First, unlike three-player games, every rational bimatrix game has a rational equilibrium. Second, a key technical step involving coloring the graphical games in the **PPAD**-hardness proofs for three/four-player games fails to extend to two-player games [18, 10, 20]. The Smoothed Lemke-Howson Conjecture was asked by a number of people [1]. Indeed, whether the smoothed analysis of the simplex algorithm can be extended to the Lemke-Howson algorithm [38] has been the question most frequently raised during talks on smoothed analysis. The conjecture is a special

case of the following conjecture posted by Spielman and Teng [55] in a survey of smoothed analysis of algorithms.

• Smoothed 2-NASH Conjecture: The smoothed complexity of BIMATRIX is polynomial.

The Smoothed 2-NASH Conjecture was inspired by the result of Bárány, Vempala and Vetta [4] that an equilibrium of a random two-player game can be found in polynomial time.

1.4 Our Contributions

Despite much effort in the last half century, no significant progress has been made in characterizing the algorithmic complexity of finding a Nash equilibrium in a two-player game. Thus, BIMATRIX, the most studied computational problem about Nash equilibria, stood out as the last open problem in equilibrium computation for normal form games. Papadimitriou [48] named it, along with FACTORING, as one of the two "most concrete open problems" at the boundary of **P**. In fact, ever since Khachiyan's discovery [34], BIMATRIX has been on the frontier of natural problems possibly solvable in polynomial time. Now, it is also on the frontier of the hard problems, assuming **PPAD** is not contained in **P**.

In this paper, we settle the computational complexity of the two-player Nash equilibrium. We prove:

Theorem 1.1. BIMATRIX is PPAD-complete.

Our result demonstrates that, even in this simplest form of non-cooperative games, equilibrium computation is polynomial-time equivalent to discrete fixed point computation. In particular, we show that from each discrete Brouwer function f, we can build a two-player game \mathcal{G} and a polynomial-time map Π from the Nash equilibria of \mathcal{G} to the fixed points of f. Our proof complements Nash's proof that for each two-player game \mathcal{G} , there is a Brouwer function f and a map Φ from the fixed points of f to the equilibrium points of \mathcal{G} .

The success in proving the **PPAD** completeness of BIMATRIX inspires us to attempt to disprove the Smoothed 2-NASH Conjecture. A connection between the smoothed complexity and approximation complexity of Nash equilibria ([55], Proposition 9.12) then leads us to prove the following result.

Theorem 1.2. For any c > 0, the problem of computing an n^{-c} -approximate Nash equilibrium of a two-player game is **PPAD**-complete.

This result enables us to establish the following fundamental theorem about the approximation of Nash equilibria. It also enables us answer the question about the smoothed complexity of the Lemke-Howson algorithm and disprove the Smoothed 2-NASH Conjecture assuming **PPAD** is not contained in **RP**.

Theorem 1.3. BIMATRIX does not have a fully polynomial-time approximation scheme unless **PPAD** is contained in **P**.

Theorem 1.4. BIMATRIX is not in smoothed polynomial time unless **PPAD** is contained in **RP**.

Consequently, it is unlikely that the $n^{O(\log n/\epsilon^2)}$ -time algorithm of Lipton, Markakis, and Mehta [40], the fastest algorithm known today for finding an ϵ -approximate Nash equilibrium, can be improved to poly $(n, 1/\epsilon)$. Also, it is unlikely that the average-case polynomial time result of [4] can be extended to the smoothed model.

Our advances in the computation, approximation, and smoothed analysis of two-player Nash equilibria are built on several novel techniques that might be interesting on their own. We introduce a new method for encoding boolean and arithmetic variables using the probability vectors of the mixed strategies. We then develop a set of perturbation techniques to simulate the boolean and arithmetic operations needed for fixed point computation using the equilibrium conditions of two-player games. These innovations enable us to bypass the graphical game model and derive a direct reduction from fixed point computation to BIMATRIX. To study the approximation and smoothed complexity of the equilibrium problem, we introduce a new discrete fixed point problem on a high-dimensional grid graph with a constant side-length. We then show that it can host the embedding of the proof structure of any **PPAD** problem. This embedding result not only enriches the family of **PPAD**-complete discrete fixed point problems, but also provides a much needed trade-off between precision and dimension. We prove a key geometric lemma for finding a high-dimensional discrete fixed point, a new concept defined on a simplex inside a unit hypercube. This geometric lemma enables us to overcome the curse of dimensionality in reasoning about fixed points in high dimensions.

1.5 Implications and Impact

Because the two-player Nash equilibrium enjoys several structural properties that Nash equilibria with three or more players do not have, our result enables us to answer some other long-standing open questions in mathematical economics and operations research. In particular, we have derived the following two important corollaries.

Corollary 1.5. Arrow-Debreu market equilibria are **PPAD**-hard to compute.

Corollary 1.6. The P-matrix Linear Complementary Problem is computationally harder than convex programming, unless **PPAD** is contained in **P**, where a P-matrix is a square matrix with positive principle minors.

To prove the first corollary, we use a recent discovery of Ye [58] (see also [15]) on the connection between two-player Nash equilibria and Arrow-Debreu equilibria in two-group Leontief exchange economies. The second corollary concerns the linear complementary problem, in which we are given a rational *n*-by-*n* matrix **M** and a rational *n*-place vector **q**, and are asked to find vectors **x** and **y** such that $\mathbf{y} = \mathbf{M}\mathbf{x} + \mathbf{q}$, $\mathbf{x}, \mathbf{y} \ge \mathbf{0}$, and $\mathbf{x}^T\mathbf{y} = 0$. Our result complements Megiddo's observation [41] that if it is **NP**-hard to solve the P-Matrix linear complementarity problem, then $\mathbf{NP} = \mathbf{coNP}$. By applying a recent reduction of Abbott, Kane, and Valiant [2], our result also implies the following corollary.

Corollary 1.7. WIN-LOSE BIMATRIX is **PPAD**-complete, where, in a win-lose bimatrix game, each payoff entry is either 0 or 1.

We further refine our reduction to show that the Nash equilibria in sparse two-player games are hard to compute and hard to approximate in fully polynomial time.

We have also discovered several new structural properties about Nash equilibria. In particular, we prove an equivalence result about various notions of approximate Nash equilibria. We exploit these equivalences in the study of the complexity of finding an approximate Nash equilibrium and in the smoothed analysis of BIMATRIX. Using them, we can also extend our result about approximate Nash equilibria as follows.

Theorem 1.8. For any c > 0, the problem of finding the first $(1 + c) \log n$ bits of an exact Nash equilibrium in a two-player game, even when the payoffs are integers of polynomial magnitude, is polynomial-time equivalent to BIMATRIX.

Recently, Chen, Teng, and Valiant [14] extended our approximation complexity result to win-lose two-player games; Huang and Teng [28] extended both the smoothed complexity and the approximation results to the computation of Arrow-Debreu equilibria. Using the connection between Nash equilibria and Arrow-Debreu equilibria, our complexity result on sparse games can be extended to market equilibria in economies with sparse exchange structures [12].

1.6 Paper Organization

In Section 2, we review concepts in equilibrium theory. We also prove an important equivalence between various notions of approximate Nash equilibria. In Section 3, we recall the complexity class **PPAD**, the smoothed analysis framework, and the concept of polynomial-time reduction among search problems. In Section 4, we introduce two concepts: high-dimensional discrete Brouwer fixed points and generalized circuits, followed by the definitions of two search problems based on these concepts. In Section 5, we state our main results and also provide an outline of our proofs. In Section 6, we show that one can simulate generalized circuits with two-player Nash equilibria. In Section 7, we prove a **PPAD**-completeness result for a large family of high-dimensional fixed point search problems. In Section 8, we complete our proof by showing that discrete fixed points can be modeled by generalized circuits. In Section 9, we discuss some extensions of our work and present several open questions and conjectures motivated by this research. In particular, we will show that sparse BIMATRIX does not have a fully polynomial-time approximation scheme unless **PPAD** is contained in **P**. Finally, in Section 10, we thank many wonderful people who helped us in this work.

This paper combines the papers "Settling the Complexity of 2-Player Nash-Equilibrium", by Xi Chen and Xiaotie Deng, and "Computing Nash Equilibria: Approximation and Smoothed Complexity", by the three of us. The extended abstracts of both papers appeared in the *Proceedings of the 47th Annual Symposium on Foundations of Computer Science, IEEE.* The result

that BIMATRIX is **PPAD**-complete is from the first paper. We also include the main result from the paper "Sparse Games are Hard", by the three of us, presented at the *the 2nd International Workshop on Internet and Network Economics*.

1.7 Notation

We will use bold lower-case Roman letters such as \mathbf{x} , \mathbf{a} , \mathbf{b}_j to denote vectors. Whenever a vector, say $\mathbf{a} \in \mathbb{R}^n$ is present, its components will be denoted by lower-case Roman letters with subscripts, such as $a_1, ..., a_n$. Matrices are denoted by bold upper-case Roman letters such as \mathbf{A} and scalars are usually denoted by lower-case roman letters, but sometimes by upper-case Roman letters such as M, N, and K. The $(i, j)^{th}$ entry of a matrix \mathbf{A} is denoted by $a_{i,j}$. Depending on the context, we may use \mathbf{a}_i to denote the i^{th} row or the i^{th} column of \mathbf{A} .

We now enumerate some other notations that are used in this paper. We will let \mathbb{Z}^d_+ to denote the set of *d*-dimensional vectors with positive integer entries; $\langle \mathbf{a} | \mathbf{b} \rangle$ to denote the dot-product of two vectors in the same dimension; \mathbf{e}_i to denote the unit vector whose i^{th} entry is equal to 1 and all other entries are 0. Finally, for $a, b \in \mathbb{R}$, by $a = b \pm \epsilon$, we mean $b - \epsilon \leq a \leq b + \epsilon$.

2 Two-Player Nash Equilibria

A two-player game [45, 37, 38] is a non-cooperative game between two players. When the first player has m choices of actions and the second player has n choices of actions, the game, in its normal form, can be specified by two $m \times n$ matrices $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$. If the first player chooses action i and the second player chooses action j, then their payoffs are $a_{i,j}$ and $b_{i,j}$, respectively. Thus, a two-player game is also often referred to as a bimatrix game. A mixed strategy of a player is a probability distribution over its choices. The Nash's Equilibrium Theorem [45, 44], when specialized to bimatrix games, asserts that every two-player game has an equilibrium point, i.e., a profile of mixed strategies, such that neither player can gain by changing his or her strategy unilaterally. The zero-sum two-player game [43] is a special case of the bimatrix game that satisfies $\mathbf{B} = -\mathbf{A}$.

Let \mathbb{P}^n denote the set of all *probability vectors* in \mathbb{R}^n , i.e., non-negative, *n*-place vectors whose entries sum to 1. Then, a profile of mixed strategies can be expressed by two column vectors $(\mathbf{x}^* \in \mathbb{P}^m, \mathbf{y}^* \in \mathbb{P}^n)$.

Mathematically, a Nash equilibrium of a bimatrix game (\mathbf{A}, \mathbf{B}) is a pair $(\mathbf{x}^* \in \mathbb{P}^m, \mathbf{y}^* \in \mathbb{P}^n)$ such that

$$(\mathbf{x}^*)^T \mathbf{A} \mathbf{y}^* \ge \mathbf{x}^T \mathbf{A} \mathbf{y}^*$$
 and $(\mathbf{x}^*)^T \mathbf{B} \mathbf{y}^* \ge (\mathbf{x}^*)^T \mathbf{B} \mathbf{y}$, for all $\mathbf{x} \in \mathbb{P}^m$ and $\mathbf{y} \in \mathbb{P}^n$.

Computationally, one might settle with an approximate Nash equilibrium. There are several versions of approximate equilibrium points that have been defined in the literature. The following are two most popular ones.

For a positive parameter ϵ , an ϵ -approximate Nash equilibrium of a bimatrix game (A, B) is

a pair $(\mathbf{x}^* \in \mathbb{P}^m, \mathbf{y}^* \in \mathbb{P}^n)$ such that

$$(\mathbf{x}^*)^T \mathbf{A} \mathbf{y}^* \ge \mathbf{x}^T \mathbf{A} \mathbf{y}^* - \epsilon \text{ and } (\mathbf{x}^*)^T \mathbf{B} \mathbf{y}^* \ge (\mathbf{x}^*)^T \mathbf{B} \mathbf{y} - \epsilon, \text{ for all } \mathbf{x} \in \mathbb{P}^m \text{ and } \mathbf{y} \in \mathbb{P}^n.$$

An ϵ -relatively-approximate Nash equilibrium of (\mathbf{A}, \mathbf{B}) is a pair $(\mathbf{x}^*, \mathbf{y}^*)$ such that

$$(\mathbf{x}^*)^T \mathbf{A} \mathbf{y}^* \ge (1-\epsilon) \mathbf{x}^T \mathbf{A} \mathbf{y}^*$$
 and $(\mathbf{x}^*)^T \mathbf{B} \mathbf{y}^* \ge (1-\epsilon) (\mathbf{x}^*)^T \mathbf{B} \mathbf{y}$, for all $\mathbf{x} \in \mathbb{P}^m$ and $\mathbf{y} \in \mathbb{P}^n$.

Nash equilibria of a bimatrix game (\mathbf{A}, \mathbf{B}) are invariant under positive scalings, meaning, the bimatrix game $(c_1\mathbf{A}, c_2\mathbf{B})$ has the same set of Nash equilibria as (\mathbf{A}, \mathbf{B}) , as long as $c_1, c_2 > 0$. They are also invariant under shifting: For any constants c_1 and c_2 , the bimatrix game $(c_1 + \mathbf{A}, c_2 + \mathbf{B})$ has the same set of Nash equilibria as (\mathbf{A}, \mathbf{B}) . It is easy to verify that ϵ -approximate Nash equilibria are also invariant under shifting. However, each ϵ -approximate Nash equilibrium (\mathbf{x}, \mathbf{y}) of (\mathbf{A}, \mathbf{B}) becomes a $(c \cdot \epsilon)$ -approximate Nash equilibria are invariant under positive scaling, but may not be invariant under shifting.

The notion of the ϵ -approximate Nash equilibrium is defined in the additive fashion. To study its complexity, it is important to consider bimatrix games with normalized matrices in which the absolute value of each entry is bounded, for example, by 1. Earlier work on this subject by Lipton, Markakis, and Mehta [40] used a similar normalization. Let $\mathbb{R}_{[a:b]}^{m \times n}$ denote the set of $m \times n$ matrices with real entries between a and b. In this paper, we say a bimatrix game (**A**, **B**) is normalized if $\mathbf{A}, \mathbf{B} \in \mathbb{R}_{[-1,1]}^{m \times n}$ and is positively normalized if $\mathbf{A}, \mathbf{B} \in \mathbb{R}_{[0,1]}^{m \times n}$.

Proposition 2.1. In a normalized two-player game (\mathbf{A}, \mathbf{B}) , every ϵ -relatively-approximate Nash equilibrium is also an ϵ -approximate Nash equilibrium.

To define our main search problems of computing and approximating a two-player Nash equilibrium, we need to first define the input models. The most general input model is the *real model* in which a bimatrix game is specified by two real matrices (**A**, **B**). In the *rational model*, each entry of the payoff matrices is given by the ratio of two integers. The *input size* is then the total number of bits describing the payoff matrices. Clearly, by multiplying the common denominators in a payoff matrix and using the fact that two-player Nash equilibria are invariant under positive scaling, we can transform a rational bimatrix game into an *integer bimatrix game*. Moreover, the total number of bits in this game with integer payoffs is within a factor of poly(m, n) of the input size of its rational counterpart. In fact, Abbott, Kane, and Valiant [2] go one step further to show that from every bimatrix game with integer payoffs, one can construct a "homomorphic" bimatrix game with 0-1 payoffs who size is within a polynomial factor of the input size of the original game.

It is well known that each rational bimatrix game has a rational Nash equilibrium. We may verify this fact as following. Suppose (\mathbf{A}, \mathbf{B}) is a rational two-player game and (\mathbf{u}, \mathbf{v}) is one of its Nash equilibria. Let row-support = $\{i \mid u_i > 0\}$ and column-support = $\{i \mid v_i > 0\}$. Let \mathbf{a}_i and \mathbf{b}_j denote the i^{th} row of \mathbf{A} and the j^{th} column of \mathbf{B} , respectively. Then, by the condition of the Nash equilibrium, (\mathbf{u}, \mathbf{v}) is a feasible solution to the following linear program:

$\sum_i x_i = 1$ and $\sum_i y_i = 1$	
$x_i = 0,$	$\forall i \notin \text{row-support}$
$y_i = 0,$	$\forall i \notin \text{column-support}$
$x_i \ge 0,$	$\forall i \in \text{row-support}$
$y_i \ge 0,$	$\forall i \in \text{column-support}$
$\mathbf{a}_i \mathbf{y} = \mathbf{a}_j \mathbf{y},$	$\forall i, j \in \text{row-support}$
$\mathbf{x}^T \mathbf{b}_i = \mathbf{x}^T \mathbf{b}_j,$	$\forall i, j \in \text{column-support}$
$\mathbf{a}_i \mathbf{y} \leq \mathbf{a}_j \mathbf{y},$	$\forall i \not\in \text{row-support}, j \in \text{row-support}$
$\mathbf{x}^T \mathbf{b}_i \leq \mathbf{x}^T \mathbf{b}_j,$	$\forall i \notin \text{column-support}, j \in \text{column-support}.$

In fact, any solution to this linear program is a Nash equilibrium of (\mathbf{A}, \mathbf{B}) . Therefore, (\mathbf{A}, \mathbf{B}) has at least one rational equilibrium point such that the total number of bits describing this equilibrium is within a polynomial factor of the input size of (\mathbf{A}, \mathbf{B}) . By enumerating all possible row supports and column supports and applying the linear program above, we can find a Nash equilibrium in the bimatrix game (\mathbf{A}, \mathbf{B}) . This exhaustive-search algorithms takes $2^{m+n} \text{poly}(L)$ time where L is the input size of the game, and m and n are, respectively, the number of rows and the number of columns.

In this paper, we use BIMATRIX to denote the problem of finding a Nash equilibrium in a rational bimatrix game. Without loss of generality, we make two assumptions about our search problem: all input bimatrix games are positively normalized in which both players have the same number of choices of actions. Thus, two important parameters associated with each instance to BIMATRIX are: n, the number of actions, and L, the total number of bits in the description of the game. Thus, BIMATRIX is in **P** if there exists an algorithm for BIMATRIX with running time poly(n, L). As a matter of fact, for the two-player games that we will design in our complexity studies, L is bounded by a polynomial in n.

We also consider two families of approximation problems for two-player Nash equilibria. For a positive constant c,

- let EXP^c-BIMATRIX denote the following search problem: Given a rational and positively normalized bimatrix game (**A**, **B**), compute a 2^{-cn} -approximate Nash equilibrium of (**A**, **B**), if **A** and **B** are $n \times n$ matrices;
- let POLY^c-BIMATRIX denote the following search problem: Given a rational and positively normalized bimatrix game (\mathbf{A}, \mathbf{B}) , compute an n^{-c} -approximate Nash equilibrium of (\mathbf{A}, \mathbf{B}) , if \mathbf{A} and \mathbf{B} are $n \times n$ matrices.

In our analysis, we will use an alternative notion of approximate Nash equilibria as introduced in [18], originally called ϵ -Nash equilibria. In order to avoid confusion with more commonly used ϵ -approximate Nash equilibria, we will refer to this alternative approximation as the ϵ -wellsupported Nash equilibrium. For a bimatrix game (**A**, **B**), let **a**_i and **b**_j denote the *i*th row of **A** and the j^{th} column of **B**, respectively. In a profile of mixed strategies (\mathbf{x}, \mathbf{y}) , the expected payoff of the first player when choosing the i^{th} row is $\mathbf{a}_i \mathbf{y}$, and the expected payoff of the second player when choosing the i^{th} column is $\mathbf{x}^T \mathbf{b}_i$.

For a positive parameter ϵ , a pair of strategies $(\mathbf{x}^* \in \mathbb{P}^n, \mathbf{y}^* \in \mathbb{P}^n)$ is an ϵ -well-supported Nash equilibrium of (\mathbf{A}, \mathbf{B}) if for all j and k,

$$(\mathbf{x}^*)^T \mathbf{b}_j > (\mathbf{x}^*)^T \mathbf{b}_k + \epsilon \Rightarrow y_k^* = 0 \text{ and } \mathbf{a}_j \mathbf{y}^* > \mathbf{a}_k \mathbf{y}^* + \epsilon \Rightarrow x_k^* = 0.$$

A Nash equilibrium is a 0-well-supported Nash equilibrium as well as a 0-approximate Nash equilibrium. The following lemma, a key lemma in our complexity study of equilibrium approximation, shows that approximate Nash equilibria and well-supported Nash equilibria are polynomially related. This polynomial relation allows us to focus our attention on pair-wise approximation conditions. Thus, we can locally argue certain properties of the bimatrix game in our analysis.

Lemma 2.2 (Polynomial Equivalence). In a bimatrix game (\mathbf{A}, \mathbf{B}) with $\mathbf{A}, \mathbf{B} \in \mathbb{R}_{[0:1]}^{n \times n}$, for any $0 \le \epsilon \le 1$,

- 1. each ϵ -well-supported Nash equilibrium is also an ϵ -approximate Nash equilibrium; and
- 2. from any $\epsilon^2/8$ -approximate Nash equilibrium (\mathbf{u}, \mathbf{v}) , one can find in polynomial time an ϵ -well-supported Nash equilibrium (\mathbf{x}, \mathbf{y}) .

Proof. The first statement follows from the definitions. Because (\mathbf{u}, \mathbf{v}) is an $\epsilon^2/8$ -approximate Nash equilibrium, we have

$$\forall \ \mathbf{u}' \in \mathbb{P}^n, \ (\mathbf{u}')^T \mathbf{A} \mathbf{v} \leq \mathbf{u}^T \mathbf{A} \mathbf{v} + \epsilon^2 / 8, \ \text{ and } \ \forall \ \mathbf{v}' \in \mathbb{P}^n, \ \mathbf{u}^T \mathbf{B} \mathbf{v}' \leq \mathbf{u}^T \mathbf{B} \mathbf{v} + \epsilon^2 / 8.$$

Recall that \mathbf{a}_i denotes the i^{th} row of \mathbf{A} and \mathbf{b}_i denotes the i^{th} column of \mathbf{B} . We use J_1 to denote the set of indices $j : 1 \leq j \leq n$ such that $\mathbf{a}_i \mathbf{v} \geq \mathbf{a}_j \mathbf{v} + \epsilon/2$, for some $i \in [1 : n]$. Let i^* be an index such that $\mathbf{a}_{i^*} \mathbf{v} = \max_{1 \leq i \leq n} \mathbf{a}_i \mathbf{v}$. Now by changing $u_j, j \in J_1$, to 0 and changing u_{i^*} to $u_{i^*} + \sum_{j \in J_1} u_j$ we can increase the first-player's profit by at least $(\epsilon/2) \sum_{j \in J_1} u_j$, implying $\sum_{j \in J_1} u_j < \epsilon/4$. Similarly, we define $J_2 = \{j : 1 \leq j \leq n : \exists i, \mathbf{u}^T \mathbf{b}_i \geq \mathbf{u}^T \mathbf{b}_j + \epsilon/2\}$. Then we have $\sum_{i \in J_2} v_j < \epsilon/4$.

We now set all these $\{u_j \mid j \in J_1\}$ and $\{v_j \mid j \in J_2\}$ to zero, and uniformly increase the probabilities of other strategies to obtain a new pair of mixed strategies (\mathbf{x}, \mathbf{y}) .

Note for all $i \in [1:n]$, $|\mathbf{a}_i \mathbf{y} - \mathbf{a}_i \mathbf{v}| \leq \epsilon/4$, because we assume the value of each entry in \mathbf{a}_i is between 0 and 1. Therefore, for every pair $i, j : 1 \leq i, j \leq n$, the relative change between $\mathbf{a}_i \mathbf{y} - \mathbf{a}_j \mathbf{y}$ and $\mathbf{a}_i \mathbf{v} - \mathbf{a}_j \mathbf{v}$ is no more than $\epsilon/2$. Thus, any j that is beaten by some i by a gap of ϵ is already set to zero in (\mathbf{x}, \mathbf{y}) .

We conclude this section by pointing out that there are other natural notions of approximation for equilibrium points. In addition to the rational representation of a rational equilibrium, one can use binary representations to define entries in an equilibrium. As each entry p in an equilibrium is a number between 0 and 1, we can specify it using its binary representation $(0 \cdot c_1 \cdots c_P \cdots)$, where $c_i \in \{0, 1\}$ and $p = \lim_{i \to \infty} \sum_{i=1} c_i/2^i$. Some rational numbers may not have a finite binary representation. Usually, we round off the numbers to store their finite approximations. The first P bits $c_1, ..., c_P$ give us a P-bit approximation \tilde{c} of c.

For a positive integer P, we will use P-BIT-BIMATRIX to denote the search problem of computing the first P bits of the entries of a Nash equilibrium in a rational bimatrix game. The following proposition relates P-BIT-BIMATRIX with POLY^c-BIMATRIX.

Proposition 2.3. Suppose (\mathbf{x}, \mathbf{y}) is a Nash equilibrium of a positively normalized two-player game (\mathbf{A}, \mathbf{B}) with *n* rows and *n* columns. For a positive integer *P*, let $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ be the *P*-bit approximation of (\mathbf{x}, \mathbf{y}) . Let $\bar{\mathbf{x}} = \tilde{\mathbf{x}} / \|\tilde{\mathbf{x}}\|_1$ and $\bar{\mathbf{y}} = \tilde{\mathbf{y}} / \|\tilde{\mathbf{y}}\|_1$. Then, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a $(3n2^{-P})$ -approximate Nash equilibrium of (\mathbf{A}, \mathbf{B}) .

Proof. A similar proposition is stated and proved in [14].

Let $a = 2^{-P}$. Suppose $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is not a (3na)-approximate Nash equilibrium. Without loss of generality, assume there exists $\mathbf{x}' \in \mathbb{P}^n$ such that $(\mathbf{x}')^T \mathbf{A} \bar{\mathbf{y}} > \bar{\mathbf{x}}^T \mathbf{A} \bar{\mathbf{y}} + 3na$. We have

$$\begin{aligned} (\mathbf{x}')^T \mathbf{A} \bar{\mathbf{y}} &\leq (\mathbf{x}')^T \mathbf{A} \tilde{\mathbf{y}} + na &\leq (\mathbf{x}')^T \mathbf{A} \mathbf{y} + na &\leq \mathbf{x}^T \mathbf{A} \mathbf{y} + na \\ &\leq \tilde{\mathbf{x}}^T \mathbf{A} \mathbf{y} + 2na &\leq \tilde{\mathbf{x}}^T \mathbf{A} \tilde{\mathbf{y}} + 3na &\leq \bar{\mathbf{x}}^T \mathbf{A} \tilde{\mathbf{y}} + 3na, \end{aligned}$$

which contradicts our assumption. To see the first inequality, note that since the game is positively normalized, every component in $(\mathbf{x}')^T \mathbf{A}$ is between 0 and 1. The inequality follows from the fact that $\bar{y}_i \geq \tilde{y}_i$ for all $i \in [1 : n]$, and $\|\tilde{\mathbf{y}}\|_1 \geq 1 - na$. The other inequalities can be proved similarly.

3 Complexity and Algorithm Analysis

In this section, we review the complexity class **PPAD** and the concept of polynomial-time reduction among search problems. We then define the perturbation models in the smoothed analysis of BIMATRIX and show that if the smoothed complexity of BIMATRIX is polynomial, then we can compute an ϵ -approximate Nash equilibrium of a bimatrix game in randomized poly $(n, 1/\epsilon)$ time.

3.1 PPAD and Polynomial-Time Reduction Among Search Problems

A binary relation $R \subset \{0,1\}^* \times \{0,1\}^*$ is polynomially balanced if there exist constants c and k such that for all pairs $(x,y) \in R$, $|y| \leq c |x|^k$, where |x| denotes the length of string x. It is polynomial-time computable if for each pair (x,y), one can decide whether or not $(x,y) \in R$ in time polynomial in |x| + |y|. One can define the **NP** search problem SEARCH^R specified by R as: Given $x \in \{0,1\}^*$, return a y such satisfying $(x,y) \in R$, if such y exists, otherwise, return a special string "no".

A relation R is *total* if for every string $x \in \{0,1\}^*$, there exists y such that $(x,y) \in R$. Following Megiddo and Papadimitriou [42], let **TFNP** denote the class of all **NP** search problems specified by total relations. A search problem SEARCH^{R_1} \in **TFNP** is *polynomial-time reducible* to problem SEARCH^{R_2} \in **TFNP** if there exists a pair of polynomial-time computable functions (f,g) such that for every x of R_1 , if y satisfies that $(f(x), y) \in R_2$, then $(x, g(y)) \in R_1$. Search problems SEARCH^{R₁} and SEARCH^{R₂} are polynomial-time equivalent if SEARCH^{R₂} is also reducible to SEARCH^{R₁}.

The complexity class **PPAD** [47] is a sub-class of **TFNP**, containing all search problems polynomial-time reducible to following problem called END-OF-LINE:

Definition 3.1 (END-OF-LINE). The input instance of END-OF-LINE is a pair $(\mathcal{M}, 0^n)$ where \mathcal{M} is a circuit of size polynomial in n that defines a function \mathcal{M} satisfying:

- for every $v \in \{0,1\}^n$, M(v) is an ordered pair (u_1, u_2) where $u_1, u_2 \in \{0,1\}^n \cup \{$ "no" $\}$.
- $M(0^n) = ("no", 1^n)$ and the first component of $M(1^n)$ is 0^n .

This instance defines a directed graph $G_M = (V, E_M)$ with $V = \{0, 1\}^n$ and $(u, v) \in E_M$, if and only if v is the second component of M(u) and u is the first component of M(v).

The output of this problem is an end vertex G_M other than 0^n , where a vertex of V is an end vertex if the summation of its in-degree and out-degree is equal to one.

Note that in graph G_M , both the in-degree and the out-degree of each vertex are at most 1. Thus, edges of G_M form a collection of directed paths and directed cycles. Because 0^n has in-degree 0 and out-degree 1, it is an end vertex in G_M . G_M must have at least one directed path. Hence, it has another end vertex and END-OF-LINE is a member of **TFNP**.

In fact, G_M has an odd number of end vertices other than 0^n . By evaluating the polynomialsized circuit \mathcal{M} on an input $v \in V$, we can access the predecessor and the successor of v.

Many important problems, such as the search versions of Brouwer's Fixed Point Theorem, Kakutani's Fixed Point Theorem, Smith's Theorem, and Borsuk-Ulam Theorem, have been shown to be in the class **PPAD** [46].

BIMATRIX is also in **PPAD** [46]. As a corollary, for all c > 0, POLY^c-BIMATRIX and EXP^c-BIMATRIX are in **PPAD**. However, it is not clear whether *P*-BIT-BIMATRIX, for a positive integer *P*, is in **PPAD**.

3.2 Smoothed Models of Bimatrix Games

In the smoothed analysis of the bimatrix game, we consider perturbed games in which each entry of the payoff matrices is subject to a small and independent random perturbation. For a pair of $n \times n$ normalized matrices $\overline{\mathbf{A}} = (\overline{a}_{i,j})$ and $\overline{\mathbf{B}} = (\overline{b}_{i,j})$, in the smoothed model, the input instance³ is then defined by (\mathbf{A}, \mathbf{B}) where $a_{i,j}$ and $b_{i,j}$ are, respectively, independent perturbations of $\overline{a}_{i,j}$ and $\overline{b}_{i,j}$ with magnitude σ .

 $^{^{3}}$ For the simplicity of presentation, in this subsection, we model the entries of payoff matrices and perturbations by real numbers. Of course, to connect with the complexity result of the previous section, where entries of matrices are in finite representations, we are mindful that some readers may prefer that we state our result and write the proof more explicitly using the finite representations. Using Equations (16) and (17) in the proof of Lemma 3.2 (see Appendix A), we can define a discrete version of the uniform and Gaussian perturbations and state and prove the same result.

There might be several models of perturbations for $a_{i,j}$ and $b_{i,j}$ with magnitude σ [55]. The two common perturbation models are the uniform perturbation and the Gaussian perturbation.

In the uniform perturbation with magnitude σ , $a_{i,j}$ and $b_{i,j}$ are chosen uniformly from the intervals $[\bar{a}_{i,j} - \sigma, \bar{a}_{i,j} + \sigma]$ and $[\bar{b}_{i,j} - \sigma, \bar{b}_{i,j} + \sigma]$, respectively. In the Gaussian perturbation with variance σ^2 , $a_{i,j}$ and $b_{i,j}$ are, respectively, chosen with density

$$\frac{1}{\sqrt{2\pi\sigma}} e^{-|a_{i,j}-\bar{a}_{i,j}|^2/2\sigma^2} \quad \text{and} \quad \frac{1}{\sqrt{2\pi\sigma}} e^{-|b_{i,j}-\bar{b}_{i,j}|^2/2\sigma^2}$$

We refer to these perturbations as σ -uniform and σ -Gaussian perturbations, respectively.

The smoothed complexity of an algorithm J for BIMATRIX is defined as following: Let $T_J(\mathbf{A}, \mathbf{B})$ be the complexity of J for finding a Nash equilibrium in a bimatrix game (\mathbf{A}, \mathbf{B}) . Then, the *smoothed complexity* of J under perturbations $N_{\sigma}()$ of magnitude σ is

Smoothed_J
$$[n, \sigma] = \max_{\bar{\mathbf{A}}, \bar{\mathbf{B}} \in \mathbb{R}_{[-1,1]}^{n \times n}} E_{\mathbf{A} \leftarrow N_{\sigma}(\bar{\mathbf{A}}), \mathbf{B} \leftarrow N_{\sigma}(\bar{\mathbf{B}})} [T_J(\mathbf{A}, \mathbf{B})],$$

where we use $\mathbf{A} \leftarrow N_{\sigma}(\bar{\mathbf{A}})$ to denote that \mathbf{A} is a perturbation of $\bar{\mathbf{A}}$ according to $N_{\sigma}(\bar{\mathbf{A}})$.

An algorithm J has a polynomial smoothed time complexity [55] if for all $0 < \sigma < 1$ and for all positive integer n, there exist positive constants c, k_1 and k_2 such that

Smoothed_J
$$[n, \sigma] \leq c \cdot n^{k_1} \sigma^{-k_2}$$
.

BIMATRIX is in *smoothed polynomial time* if there exists an algorithm J with polynomial smoothed time complexity for computing a two-player Nash equilibrium.

The following lemma shows that if the smoothed complexity of BIMATRIX is low, under uniform or Gaussian perturbations, then one can quickly find an approximate Nash equilibrium.

Lemma 3.2 (Smoothed Nash vs Approximate Nash). If BIMATRIX is in smoothed polynomial time under uniform or Gaussian perturbations, then for all $\epsilon > 0$, there exists a randomized algorithm to compute an ϵ -approximate Nash equilibrium in a two-player game with expected time $O(\operatorname{poly}(m, n, 1/\epsilon))$ or $O(\operatorname{poly}(m, n, \sqrt{\log \max(m, n)}/\epsilon))$, respectively.

Proof. Informally argued in [55]. See Appendix A for a proof.

4 Two Search Problems

In this section, we consider two search problems that are essential to our main results. In the first problem, the objective is to find a high-dimensional discrete Brouwer fixed point. To define the second problem, we introduce a concept of the generalized circuit.

4.1 Discrete Brouwer Fixed Points

The following is an oblivious fact: Suppose we color the endpoints of an interval [0, n] by two distinct colors, say red and blue, insert n - 1 points evenly into this interval to subdivide it into

n unit subintervals, and color these new points arbitrarily by one of the two colors. Then, there must be a *bichromatic subinterval*, i.e., an unit subinterval whose two endpoints have distinct colors.

Our first search problem is built on a high-dimensional extension of this fact. Instead of coloring points in a subdivision of an intervals, we color the vertices in a hypergrid. If the dimension is d, we will use d + 1 colors.

For $d \in \mathbb{Z}^1_+$ and \mathbf{r} in \mathbb{Z}^d_+ , let $A^d_{\mathbf{r}} = \{\mathbf{q} \in \mathbb{Z}^d \mid 0 \le q_i \le r_i - 1, \forall i \in [1:d]\}$ denote the vertices of the *hypergrid* with side lengths specified by \mathbf{r} . The *boundary* of $A^d_{\mathbf{r}}$, $\partial(A^d_{\mathbf{r}})$, is the set of points $\mathbf{q} \in A^d_{\mathbf{r}}$ with $q_i \in \{0, r_i - 1\}$ for some *i*. Let Size $[\mathbf{r}] = \sum_{1 \le i \le d} \lceil \log(r_i + 1) \rceil$.

In one dimension, the interval [0, n] is the union of n unit subintervals. A hypergrid can be viewed as the union of a collection of unit hypercubes. For a point $\mathbf{p} \in \mathbb{Z}^d$, let $K_{\mathbf{p}} = \{\mathbf{q} \in \mathbb{Z}^d \mid q_i \in \{p_i, p_i + 1\}, \forall i \in [1 : d]\}$ be the vertices of the unit hypercube with \mathbf{p} as its corner closest to the origin.

We can color the vertices of a hypergrid with (d + 1) colors $\{1, 2, ..., d + 1\}$. Like in one dimension, the coloring of the boundary vertices needs to meet certain requirements in the context of the discrete Brouwer fixed point problem. A color assignment ϕ of $A_{\mathbf{r}}^d$ is valid if $\phi(\mathbf{p})$ satisfies the following condition: For $\mathbf{p} \in \partial(A_{\mathbf{r}}^d)$, if there exists an $i \in [1 : d]$ such that $p_i = 0$ then $\phi(\mathbf{p}) = \max\{i \mid p_i = 0\}$; otherwise $\phi(\mathbf{p}) = d + 1$. In the later case, $\forall i, p_i \neq 0$ and $\exists i, p_i = r_i - 1$.

The following theorem is a high-dimensional extension of the one-dimensional fact mentioned above. It is also an extension of the two-dimensional Sperner's Lemma.

Theorem 4.1 (High-Dimensional Discrete Brouwer Fixed Points). For $d \in \mathbb{Z}^1_+$ and \mathbf{r} in \mathbb{Z}^d_+ , for any valid coloring ϕ of $\mathbf{A}^d_{\mathbf{p}}$, there is a unit hypercube in $\mathbf{A}^d_{\mathbf{p}}$ whose vertices have all d + 1 colors.

In other words, Theorem 4.1 asserts that there exists a $\mathbf{p} \in A^d_{\mathbf{r}}$ such that ϕ assigns all (d+1) colors to $K_{\mathbf{p}}$. We call $K_{\mathbf{p}}$ a panchromatic cube. However, in d-dimensions, a panchromatic cube contains 2^d vertices. This exponential dependency in the dimension makes it inefficient to check whether a hypercube is panchromatic. We introduce the following notion of discrete fixed points.

Definition 4.2 (Panchromatic Simplex). A subset $P \subset A_{\mathbf{r}}^d$ is accommodated if $P \subset K_{\mathbf{p}}$ for some point $\mathbf{p} \in A_{\mathbf{r}}^d$. $P \subset A_{\mathbf{r}}^d$ is a panchromatic simplex of a color assignment ϕ if it is accommodated and contains exactly d + 1 points with d + 1 distinct colors.

Corollary 4.3 (Existence of Panchromatic Simplex). For $d \in \mathbb{Z}^1_+$ and \mathbf{r} in \mathbb{Z}^d_+ , for any valid coloring ϕ of $\mathbf{A}^d_{\mathbf{p}}$, there exists a panchromatic simplex in $\mathbf{A}^d_{\mathbf{p}}$.

We can define a search problem based on Theorem 4.1, or precisely, based on Corollary 4.3. An input instance is a hypergrid together with a polynomial-sized circuit for coloring the vertices of the hypergrid.

Definition 4.4 (Brouwer-Mapping Circuit and Color Assignment). For $d \in \mathbb{Z}^1_+$ and $\mathbf{r} \in \mathbb{Z}^d_+$, a Boolean circuit C with Size $[\mathbf{r}]$ input bits and 2d output bits $\Delta^+_1, \Delta^-_1, ..., \Delta^+_d, \Delta^-_d$ is a valid Brouwer-mapping circuit (with parameters d and \mathbf{r}) if the following is true.

- For every $\mathbf{p} \in A^d_{\mathbf{r}}$, the 2d output bits of C evaluated at \mathbf{p} satisfy one of the following (d+1) cases:
 - Case i, $1 \leq i \leq d$: $\Delta_i^+ = 1$ and all other 2d 1 bits are 0;
 - Case (d+1): $\forall i, \Delta_i^+ = 0$ and $\Delta_i^- = 1$.
- For every $\mathbf{p} \in \partial(A_{\mathbf{r}}^d)$, if there exists an $i \in [1 : d]$ such that $p_i = 0$, letting $i_{\max} = \max\{i \mid p_i = 0\}$, then the output bits satisfy Case i_{\max} , otherwise $(\forall i, p_i \neq 0 \text{ and } \exists i, p_i = r_i 1)$, the output bits satisfy Case d + 1.

The circuit C defines a valid color assignment $Color_C : A^d_{\mathbf{r}} \to \{1, 2, ..., d+1\}$ by setting $Color_C [\mathbf{p}] = i$, if the output bits of C evaluated at \mathbf{p} satisfy Case i.

To define our high-dimensional Brouwer's fixed point problems, we need a notion of *well-behaved* functions (please note that this is not the function for the fixed point problem) to parameterize the shape of the search space. An integer function f(n) is called *well-behaved* if it is polynomial-time computable and there exists an integer constant n_0 such that $3 \le f(n) \le n/2$ for all $n \ge n_0$. For example, $f_1(n) = 3$, $f_2(n) = \lfloor n/2 \rfloor$, $f_3(n) = \lfloor n/3 \rfloor$, and $f_4(n) = \lfloor \log n \rfloor$ are all well-behaved.

Definition 4.5 (BROUWER^f). For each well-defined function f, the search problem BROUWER^f is defined as following: Given an input instance of BROUWER^f, $(C, 0^n)$, where C is a valid Brouwer-mapping circuit with parameters $d = \lceil n/f(n) \rceil$ and $\mathbf{r} \in \mathbb{Z}^d_+$ where $\forall i \in [1:d], r_i = 2^{f(n)}$, find a panchromatic simplex of C.

The *input size* of BROUWER^f is the sum of n and the size of the circuit C. BROUWER^{f₂} is a two-dimensional search problem over grid $[0:2^{\lfloor n/2 \rfloor}-1]^2$ and BROUWER^{f₃} is a three-dimensional search problem over grid $[0:2^{\lfloor n/3 \rfloor}-1]^3$, while BROUWER^{f₁} is a $\lceil n/3 \rceil$ -dimensional search problem over grid $[0:7]^{\lceil n/3 \rceil}$. Each of these three grids contains about 2^n hypercubes. Both BROUWER^{f₂} [9] and BROUWER^{f₃} [18] are known to be **PPAD**-complete. In section 7, we will prove the following theorem, which states that the complexity of finding a panchromatic simplex is essentially independent of the shape or dimension of the search space. In particular, it implies that BROUWER^{f₁} is also **PPAD**-complete.

Theorem 4.6 (High-Dimensional Discrete Fixed Points). For each well-behaved function f, BROUWER^f is **PPAD**-complete.

4.2 Generalized Circuits and Their Assignment Problem

To effectively connect discrete Brouwer fixed points with two-player Nash equilibria, we use an intermediate structure called the generalized circuit. This family of circuits, motivated by the reduction of [18, 10, 20], extends the standard classes of Boolean or Arithmetic circuits in several aspects.

Syntactically, a generalized circuit S = (V, T) is a pair, where V is a set of nodes and T is a collection of gates. Every gate $T \in T$ is a 5-tuple $T = (G, v_1, v_2, v, \alpha)$ in which



Figure 1: An example of generalized circuits

- $G \in \{G_{\zeta}, G_{\times\zeta}, G_{=}, G_{+}, G_{-}, G_{\langle}, G_{\wedge}, G_{\vee}, G_{\neg}\}$ is the type of the gate;
- $v_1, v_2 \in V \cup \{nil\}$ are the first and second input nodes of the gate;
- $v \in V$ is the output node, and $\alpha \in \mathbb{R} \cup \{nil\}$.

The collection \mathcal{T} of gates must satisfy the following property: For every two gates $T = (G, v_1, v_2, v, \alpha)$ and $T' = (G', v'_1, v'_2, v', \alpha')$ in $\mathcal{T}, v \neq v'$.

Suppose $T = (G, v_1, v_2, v, \alpha)$ in \mathcal{T} . If $G = G_{\zeta}$, then the gate has no input node and $v_1 = v_2 = nil$. If $G \in \{G_{\times \zeta}, G_{=}, G_{\neg}\}$, then $v_1 \in V$ and $v_2 = nil$. If $G \in \{G_+, G_-, G_{<}, G_{\wedge}, G_{\vee}\}$, then $v_1, v_2 \in V$ and $v_1 \neq v_2$. Parameter α is only used in G_{ζ} and $G_{\times \zeta}$ gates. If $G = G_{\zeta}$, then $\alpha \in \mathbb{R}$ and $0 \leq \alpha \leq 1/|V|$. If $G = G_{\times \zeta}$, then $0 \leq \alpha \leq 1$. For other types of gates, $\alpha = nil$.

The *input size* of a generalized circuit is the sum of |V| and the total number of bits needed to specify the α parameters in $S = (V, \mathcal{T})$. As an important point which will become clear later, we make the following remark: In all generalized circuits that we will construct, the number of bits of each α parameter is upper bounded by poly(|V|).

In addition to its more expanded list of gate types, the generalized circuit differs crucially from the standard circuit in that it does not require the circuit to be acyclic. In other words, in a generalized circuit, the directed graph defined by connecting input nodes of all gates to their output counterparts may have cycles. We shall show later that the presence of cycles is necessary and sufficient to express fixed point computations with generalized circuits.

Semantically, we associate every node $v \in V$ with a real variable $\mathbf{x}[v]$. Each gate $T \in \mathcal{T}$ requires that the variables of its input and output nodes satisfy certain constraints, either arithmetic or logical, depending on the type of the gate. By setting $\epsilon = 0$, the constraints are defined in Figure 2. The notation $=_B^{\epsilon}$ in Figure 2 will be defined shortly. A generalized circuit defines a set of K = |V| constraints, or a mathematical program, over the set of variables $\{\mathbf{x}[v] \mid v \in V\}$.

Suppose S = (V, T) is a generalized circuit and K = |V|. For every $\epsilon \ge 0$, an ϵ -approximate solution to circuit S is an assignment to the variables $\{\mathbf{x}[v] \mid v \in V\}$ such that

• the values of **x** satisfy constraint $\mathcal{P}[\epsilon] = \left[0 \leq \mathbf{x}[v] \leq 1/K + \epsilon, \forall v \in V \right]$; and

$$\begin{split} G &= G_{\zeta} \colon \quad \mathcal{P}[T,\epsilon] = \left[\begin{array}{c} \mathbf{x}[v] = \alpha \pm \epsilon \end{array} \right] \\ G &= G_{\times\zeta} \colon \quad \mathcal{P}[T,\epsilon] = \left[\begin{array}{c} \mathbf{x}[v] = \min\left(\alpha \mathbf{x}[v_1], 1/K\right) \pm \epsilon \end{array} \right] \\ G &= G_{=} \colon \quad \mathcal{P}[T,\epsilon] = \left[\begin{array}{c} \mathbf{x}[v] = \min\left(\mathbf{x}[v_1], 1/K\right) \pm \epsilon \end{array} \right] \\ G &= G_{+} \colon \quad \mathcal{P}[T,\epsilon] = \left[\begin{array}{c} \mathbf{x}[v] = \min\left(\mathbf{x}[v_1] + \mathbf{x}[v_2], 1/K\right) \pm \epsilon \end{array} \right] \\ G &= G_{-} \colon \quad \mathcal{P}[T,\epsilon] = \left[\begin{array}{c} \min\left(\mathbf{x}[v_1] - \mathbf{x}[v_2], 1/K\right) - \epsilon \leq \mathbf{x}[v] \leq \max\left(\mathbf{x}[v_1] - \mathbf{x}[v_2], 0\right) + \epsilon \end{array} \right] \\ G &= G_{-} \colon \quad \mathcal{P}[T,\epsilon] = \left[\begin{array}{c} \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] < \mathbf{x}[v_2] - \epsilon; \mathbf{x}[v] = \stackrel{\epsilon}{B} 0 \text{ if } \mathbf{x}[v_1] > \mathbf{x}[v_2] + \epsilon \end{array} \right] \\ G &= G_{\vee} \colon \quad \mathcal{P}[T,\epsilon] = \left[\begin{array}{c} \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \text{ or } \mathbf{x}[v_2] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 0 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 0 \text{ and } \mathbf{x}[v_2] = \stackrel{\epsilon}{B} 0 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \text{ and } \mathbf{x}[v_2] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 0 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 0 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \text{ and } \mathbf{x}[v_2] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 0 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 0 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 0 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \text{ if } \mathbf{x}[v_1] = \stackrel{\epsilon}{B} 1 \\ \mathbf{x}[v] = \stackrel{\epsilon}{B} 1 \\$$

Figure 2: Constraints $\mathcal{P}[T, \epsilon]$, where $T = (G, v_1, v_2, v, \alpha)$

• for each gate $T = (G, v_1, v_2, v, \alpha) \in \mathcal{T}$, the values of $\mathbf{x}[v_1], \mathbf{x}[v_2]$ and $\mathbf{x}[v]$ satisfy the constraint $\mathcal{P}[T, \epsilon]$, defined in Figure 2.

Among the nine types of gates, $G_{\zeta}, G_{\times\zeta}, G_{=}, G_{+}$ and G_{-} are arithmetic gates implementing arithmetic constraints like addition, subtraction and constant multiplication. $G_{<}$ is a *brittle* comparator; it only distinguishes values that are properly separated. Finally, G_{\wedge}, G_{\vee} and G_{\neg} are logic gates. For an assignment to variables $\{\mathbf{x}[v] \mid v \in V\}$, the value of $\mathbf{x}[v]$ represents boolean 1 with precision ϵ , denoted by $\mathbf{x}[v] =_B^{\epsilon} 1$, if $1/K - \epsilon \leq \mathbf{x}[v] \leq 1/K + \epsilon$; it represents boolean 0 with precision ϵ , denoted by $\mathbf{x}[v] =_B^{\epsilon} 0$, if $0 \leq \mathbf{x}[v] \leq \epsilon$. We will use $\mathbf{x}[v] = 1/K \pm \epsilon$ to denote the constraint that the value of $\mathbf{x}[v]$ lies in $[1/K - \epsilon, 1/K + \epsilon]$. The logic constraints implemented by the three logic gates are defined similarly as the classical ones.

From the reduction in Section 6, we can prove the following theorem. A proof can be found in **Appendix B**.

Theorem 4.7. For any constant c > 0, every generalized circuit S = (V, T) has a $1/|V|^c$ -approximate solution.

Let c be a positive constant. We use POLY^c-GCIRCUIT, and EXP^c-GCIRCUIT to denote the problems of finding a K^{-c} -approximate solution and a 2^{-cK} -approximate solution, respectively, of a given generalized circuit with K nodes.

5 Main Results and Proof Outline

As the main technical result of our paper, we prove the following theorem.

Theorem 5.1 (Main). For any constant c > 0, POLY^c-BIMATRIX is **PPAD**-complete.

This theorem immediately implies the following statements about the complexity of computing and approximating two-player Nash equilibria.

Theorem 5.2 (Complexities of BIMATRIX). BIMATRIX is **PPAD**-complete. Moreover, it does not have a fully-polynomial-time approximation scheme, unless **PPAD** is contained in **P**.

By Proposition 2.1, BIMATRIX does not have a fully polynomial-time approximation scheme in the relative approximation of Nash equilibria.

Setting $\epsilon = 1/\text{poly}(n)$, by Theorem 5.1 and Lemma 3.2, we obtain following theorem on the smoothed complexity of two-player Nash equilibria:

Theorem 5.3 (Smoothed Complexity of BIMATRIX). BIMATRIX is not in smoothed polynomial time, under uniform or Gaussian perturbations, unless **PPAD** is contained in **RP**.

Corollary 5.4 (Smoothed Complexity of Lemke-Howson). If **PPAD** is not contained in **RP**, then the smoothed complexity of the Lemke-Howson algorithm is not polynomial.

By Proposition 2.3, we obtain the following corollary from Theorem 5.1 about the complexity of BIT-BIMATRIX.

Corollary 5.5 (BIT-BIMATRIX). For any constant c > 1, $(c \log n)$ -BIT-BIMATRIX, the problem of finding the first $c \log n$ bits of a Nash equilibrium in a bimatrix game is polynomial-time equivalent to BIMATRIX.

To prove Theorem 5.1, we will start with the discrete fixed point problem BROUWER^{f_1} (recall that $f_1(n) = 3$ for all n). As f_1 is a well-behaved function, Theorem 4.6 implies that BROUWER^{f_1} is a **PPAD**-complete problem. We then apply the following three lemmas to reduce BROUWER^{f_1} to POLY^c-BIMATRIX.

Lemma 5.6 (FPC to GCIRCUIT). BROUWER^{f_1} is polynomial-time reducible to POLY³-GCIRCUIT.

Lemma 5.7 (GCIRCUIT to BIMATRIX). POLY³-GCIRCUIT *is polynomial-time reducible to* POLY¹²-BIMATRIX.

Lemma 5.8 (Padding Bimatrix Games). If POLY^c-BIMATRIX is **PPAD**-complete for some constant c > 0, then POLY^{c'}-BIMATRIX is **PPAD**-complete for every constant c' > 0.

We will prove Lemma 5.6 and Lemma 5.7, respectively, in Section 8 and Section 6. A proof of Lemma 5.8 can be found in Appendix C.

6 Simulating Generalized Circuits with Nash Equilibria

In this section, we reduce $POLY^3$ -GCIRCUIT, the problem of computing a $1/K^3$ -approximate solution of a generalized circuit of K nodes, to $POLY^{12}$ -BIMATRIX. As every two-player game has a Nash equilibrium, this reduction also implies that every generalized circuit with K nodes has a $1/K^3$ -approximate solution.

6.1 Outline of the Reduction

Suppose $S = (V, \mathcal{T})$ is a generalized circuit. Let K = |V| and N = 2K. Let \mathcal{C} be a one-toone map from V to $\{1, 3, ..., 2K - 3, 2K - 1\}$. From every vector $\mathbf{x} \in \mathbb{R}^N$, we define two maps $\overline{\mathbf{x}}, \overline{\mathbf{x}}_C : V \to \mathbb{R}$: For every node $v \in V$, supposing $\mathcal{C}(v) = 2k - 1$, we set $\overline{\mathbf{x}}[v] = x_{2k-1}$ and $\overline{\mathbf{x}}_C[v] = x_{2k-1} + x_{2k}$.

In our reduction, we will build an $N \times N$ bimatrix game $\mathcal{G}^{\mathcal{S}} = (\mathbf{A}^{\mathcal{S}}, \mathbf{B}^{\mathcal{S}})$. Our construction will take polynomial time and ensure the following properties for $\epsilon = 1/K^3$.

- Property \mathbf{A}_1 : $|a_{i,j}^{\mathcal{S}}|, |b_{i,j}^{\mathcal{S}}| \leq N^3$, for all $i, j : 1 \leq i, j \leq N$ and
- **Property** \mathbf{A}_2 : for every ϵ -well-supported Nash equilibrium (\mathbf{x}, \mathbf{y}) of game $\mathcal{G}^{\mathcal{S}}$, $\overline{\mathbf{x}}$ is an ϵ -approximate solution to \mathcal{S} .

Then, we normalize $\mathcal{G}^{\mathcal{S}}$ to obtain $\overline{\mathcal{G}^{\mathcal{S}}} = (\overline{\mathbf{A}^{\mathcal{S}}}, \overline{\mathbf{B}^{\mathcal{S}}})$ by setting

$$\overline{a^{\mathcal{S}}}_{i,j} = \frac{a^{\mathcal{S}}_{i,j} + N^3}{2N^3} \quad \text{and} \quad \overline{b^{\mathcal{S}}}_{i,j} = \frac{b^{\mathcal{S}}_{i,j} + N^3}{2N^3}, \quad \text{for all } i,j: 1 \le i,j \le N.$$

Property \mathbf{A}_2 implies that for every $\epsilon/(2N^3)$ -well-supported equilibrium (\mathbf{x}, \mathbf{y}) of $\overline{\mathcal{G}^S}$, $\overline{\mathbf{x}}$ is an ϵ -approximate solution to \mathcal{S} . By Lemma 2.2, from every $2/N^{12}$ -approximate Nash equilibrium of $\overline{\mathcal{G}^S}$, we can compute an ϵ -approximate solution to \mathcal{S} in polynomial time.

In the remainder of this section, we assume $\epsilon = 1/K^3$.

6.2 Construction of Game $\mathcal{G}^{\mathcal{S}}$

To construct $\mathcal{G}^{\mathcal{S}}$, we transform a prototype game $\mathcal{G}^* = (\mathbf{A}^*, \mathbf{B}^*)$, an $N \times N$ zero-sum game to be defined in Section 6.3, by adding $|\mathcal{T}|$ carefully designed "gadget" games: For each gate $T \in \mathcal{T}$, we define a pair of $N \times N$ matrices $(\mathbf{L}[T], \mathbf{R}[T])$, according to Figure 3. Then, we set

$$\mathcal{G}^{\mathcal{S}} = (\mathbf{A}^{\mathcal{S}}, \mathbf{B}^{\mathcal{S}}), \text{ where } \mathbf{A}^{\mathcal{S}} = \mathbf{A}^* + \sum_{T \in \mathcal{T}} \mathbf{L}[T] \text{ and } \mathbf{B}^{\mathcal{S}} = \mathbf{B}^* + \sum_{T \in \mathcal{T}} \mathbf{R}[T]$$

For each gate $T \in \mathcal{T}$, $\mathbf{L}[T]$ and $\mathbf{R}[T]$ defined in Figure 3 satisfy the following property.

Property 1. Let $T = (G, v_1, v_2, v, \alpha)$, $\mathbf{L}[T] = (L_{i,j})$ and $\mathbf{R}[T] = (R_{i,j})$. Suppose $\mathcal{C}(v) = 2k - 1$.

 $\mathbf{L}[T]$ and $\mathbf{R}[T]$, where gate $T = (G, v_1, v_2, v, \alpha)$

Set
$$\mathbf{L}[T] = (L_{i,j}) = \mathbf{R}[T] = (R_{i,j}) = 0, \ k = \mathcal{C}(v), \ k_1 = \mathcal{C}(v_1) \ \text{and} \ k_2 = \mathcal{C}(v_2)$$

 $G_+: \ L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_{1-1},2k-1} = R_{2k_{2-1},2k-1} = R_{2k-1,2k} = 1.$
 $G_{\zeta}: \ L_{2k-1,2k} = L_{2k,2k-1} = R_{2k-1,2k-1} = 1, \ R_{i,2k} = \alpha, \forall i : 1 \le i \le 2K.$
 $G_{\times\zeta}: \ L_{2k-1,2k-1} = L_{2k,2k} = R_{2k-1,2k} = 1, \ R_{2k_{1}-1,2k-1} = \alpha.$
 $G_{=}: \ L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_{1}-1,2k-1} = R_{2k_{2}-1,2k} = 1.$
 $G_{-}: \ L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_{1}-1,2k-1} = R_{2k_{2}-1,2k} = R_{2k-1,2k} = 1.$
 $G_{<}: \ L_{2k-1,2k} = L_{2k,2k-1} = R_{2k_{1}-1,2k-1} = R_{2k_{2}-1,2k} = 1.$
 $G_{\vee}: \ L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_{1}-1,2k-1} = R_{2k_{2}-1,2k-1} = 1, \ R_{i,2k} = 1/(2K), \forall i : 1 \le i \le 2K.$
 $G_{\wedge}: \ L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_{1}-1,2k-1} = R_{2k_{2}-1,2k-1} = 1, \ R_{i,2k} = 3/(2K), \forall i : 1 \le i \le 2K.$
 $G_{\neg}: \ L_{2k-1,2k} = L_{2k,2k-1} = R_{2k_{1}-1,2k-1} = R_{2k_{2}-1,2k-1} = 1.$

Figure 3: Matrices $\mathbf{L}[T]$ and $\mathbf{R}[T]$

Then,

$$\begin{split} i \not\in \{2k, 2k-1\} &\Rightarrow L_{i,j} = 0, \quad \forall \ j \in [1:2K]; \\ j \not\in \{2k, 2k-1\} &\Rightarrow R_{i,j} = 0, \quad \forall \ i \in [1:2K]; \\ i \in \{2k, 2k-1\} &\Rightarrow 0 \le L_{i,j} \le 1, \quad \forall \ j \in [1:2K]; \\ j \in \{2k, 2k-1\} &\Rightarrow 0 \le R_{i,j} \le 1, \quad \forall \ i \in [1:2K]. \end{split}$$

6.3 The Prototype Game and Its Properties

The prototype $\mathcal{G}^* = (\mathbf{A}^*, \mathbf{B}^*)$ is the bimatrix game called *Generalized Matching Pennies* with parameter $M = 2K^3$:

$$\mathbf{A}^* = \begin{pmatrix} M & M & 0 & 0 & \cdots & 0 & 0 \\ M & M & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & M & M & \cdots & 0 & 0 \\ 0 & 0 & M & M & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & M & M \\ 0 & 0 & 0 & 0 & \cdots & M & M \end{pmatrix}$$

 \mathbf{A}^* is a $K \times K$ block-diagonal matrix where each diagonal block is a 2×2 matrix of all M's, and $\mathbf{B}^* = -\mathbf{A}^*$. All games we will consider below belong to the following class:

Definition 6.1 (Class \mathcal{L}). A bimatrix game (\mathbf{A}, \mathbf{B}) is a member of \mathcal{L} if the entries in $\mathbf{A} - \mathbf{A}^*$ and $\mathbf{B} - \mathbf{B}^*$ are in [0:1]. Note that every Nash equilibrium (\mathbf{x}, \mathbf{y}) of \mathcal{G}^* enjoys the following nice property: For all $v \in V$, $\overline{\mathbf{x}}_C[v] = \overline{\mathbf{y}}_C[v] = 1/K$. We first prove an extension of this property for bimatrix games in \mathcal{L} . Recall $\epsilon = 1/K^3$.

Lemma 6.2 (Nearly Uniform Capacities). For every bimatrix game $(\mathbf{A}, \mathbf{B}) \in \mathcal{L}$, if (\mathbf{x}, \mathbf{y}) is a 1.0-well-supported Nash equilibrium of (\mathbf{A}, \mathbf{B}) , then

$$1/K - \epsilon \leq \overline{\mathbf{x}}_C[v], \overline{\mathbf{y}}_C[v] \leq 1/K + \epsilon, \text{ for all } v \in V.$$

Proof. Recall that $\langle \mathbf{a} | \mathbf{b} \rangle$ denotes the inner product of two vectors \mathbf{a} and \mathbf{b} of the same length. By the definition of class \mathcal{L} , for each k, the $2k - 1^{st}$ and $2k^{th}$ entries of rows \mathbf{a}_{2k-1} and \mathbf{a}_{2k} in \mathbf{A} are in [M, M + 1] and all other entries in these two rows are in [0, 1]. Thus, for any probability vector $\mathbf{y} \in \mathbb{P}^n$ and for each node $v \in V$, supposing $\mathcal{C}(v) = 2k - 1$, we have

$$M\overline{\mathbf{y}}_C[v] \le \langle \mathbf{a}_{2k-1} | \mathbf{y} \rangle, \langle \mathbf{a}_{2k} | \mathbf{y} \rangle \le M\overline{\mathbf{y}}_C[v] + 1.$$
 (1)

Similarly, the $(2l-1)^{th}$ and $2l^{th}$ entries of columns \mathbf{b}_{2l-1} and \mathbf{b}_{2l} in \mathbf{B} are in [-M, -M+1] and all other entries in these two columns are in [0, 1]. Thus, for any probability vector $\mathbf{x} \in \mathbb{P}^n$ and for each node $v \in V$, supposing $\mathcal{C}(v) = 2l - 1$, we have

$$-M\overline{\mathbf{x}}_{C}[v] \leq \langle \mathbf{b}_{2l-1} | \mathbf{x} \rangle, \, \langle \mathbf{b}_{2l} | \mathbf{x} \rangle \leq -M\overline{\mathbf{x}}_{C}[v] + 1.$$
⁽²⁾

Now, suppose (\mathbf{x}, \mathbf{y}) is a *t*-well-supported Nash equilibrium of (\mathbf{A}, \mathbf{B}) for $t \leq 1$. To warm up, we first prove that for each node $v \in V$, if $\overline{\mathbf{y}}_C[v] = 0$ then $\overline{\mathbf{x}}_C[v] = 0$. Note that $\overline{\mathbf{y}}_C[v] = 0$ implies there exists $v' \in V$ with $\overline{\mathbf{y}}_C[v'] \geq 1/K$. Suppose $\mathcal{C}(v) = 2l - 1$ and $\mathcal{C}(v') = 2k - 1$. By Inequality (1),

$$\langle \mathbf{a}_{2k} | \mathbf{y} \rangle - \max\left(\langle \mathbf{a}_{2l} | \mathbf{y} \rangle, \langle \mathbf{a}_{2l-1} | \mathbf{y} \rangle \right) \ge M \overline{\mathbf{y}}_C[v'] - \left(M \overline{\mathbf{y}}_C[v] + 1 \right) \ge M/K - 1 > 1$$

In other words, the payoff of the first player P_1 when choosing the $2k^{th}$ row is more than 1 plus the payoff of P_1 when choosing the $2l^{th}$ or the $(2l-1)^{th}$ row. Because (\mathbf{x}, \mathbf{y}) is a *t*-well-supported Nash equilibrium with $t \leq 1$, we have $\overline{\mathbf{x}}_C[v] = 0$.

Next, we prove $|\overline{\mathbf{x}}_C[v] - 1/K| < \epsilon$ for all $v \in V$. To derive a contradiction, we assume that this statement is not true. Then, there exist $v, v' \in V$ such that $\overline{\mathbf{x}}_C[v] - \overline{\mathbf{x}}_C[v'] > \epsilon$. Suppose $\mathcal{C}(v) = 2l - 1$ and $\mathcal{C}(v') = 2k - 1$. By Inequality (2),

$$\langle \mathbf{b}_{2k} | \mathbf{x} \rangle - \max\left(\langle \mathbf{b}_{2l} | \mathbf{x} \rangle, \langle \mathbf{b}_{2l-1} | \mathbf{x} \rangle \right) \ge -M \overline{\mathbf{x}}_C[v'] - \left(-M \overline{\mathbf{x}}_C[v] + 1 \right) > 1,$$

since $M = 2K^3 = 2/\epsilon$. This would imply $\overline{\mathbf{y}}_C[v] = 0$, and in turn imply $\overline{\mathbf{x}}_C[v] = 0$, contradicting our assumption that $\overline{\mathbf{x}}_C[v] > \overline{\mathbf{x}}_C[v'] + \epsilon > 0$.

We can similarly show $|\overline{\mathbf{y}}_C[v] - 1/K| < \epsilon$ for all $v \in V$.

6.4 Correctness of the Reduction

We now prove that, for every ϵ -well-supported equilibrium (\mathbf{x}, \mathbf{y}) of $\mathcal{G}^{\mathcal{S}}$, $\overline{\mathbf{x}}$ is an ϵ -approximate solution to $\mathcal{S} = (V, \mathcal{T})$. It suffices to show, to be accomplished by the next two lemmas, that $\overline{\mathbf{x}}$ satisfies the following collection of $1 + |\mathcal{T}|$ constraints.

$$\Big\{ \mathcal{P}[\epsilon], \text{ and } \mathcal{P}[T,\epsilon], T \in \mathcal{T} \Big\}.$$

Lemma 6.3 (Constraint $\mathcal{P}[\epsilon]$). Bimatrix game $\mathcal{G}^{\mathcal{S}}$ is in \mathcal{L} . Thus, for every ϵ -well-supported Nash equilibrium (\mathbf{x}, \mathbf{y}) of $\mathcal{G}^{\mathcal{S}}$, $\overline{\mathbf{x}}$ satisfies constraint $\mathcal{P}[\epsilon] = [0 \leq \overline{\mathbf{x}}[v] \leq 1/K + \epsilon, \forall v \in V]$.

Proof. For each gate $T \in \mathcal{T}$, $\mathbf{L}[T]$ and $\mathbf{R}[T]$) defined in Figure 3 satisfy Property 1. By the definition the generalized circuit, gates in \mathcal{T} have distinct output nodes, so $\mathcal{G}^{\mathcal{S}} \in \mathcal{L}$. The second statement of the lemma then follows from Lemma 6.2.

Lemma 6.4 (Constraints $P[T, \epsilon]$). Let (\mathbf{x}, \mathbf{y}) be an ϵ -well-supported Nash equilibrium of $\mathcal{G}^{\mathcal{S}}$. Then, for each gate $T \in \mathcal{T}$, $\overline{\mathbf{x}}$ satisfies constraint $\mathcal{P}[T, \epsilon]$.

Proof. Recall $\mathcal{P}[T, \epsilon]$ is a constraint defined in Figure 2. By Lemma 6.3, **x** and **y** satisfy

$$1/K - \epsilon \leq \overline{\mathbf{x}}_C[v], \, \overline{\mathbf{y}}_C[v] \leq 1/K + \epsilon, \text{ for all } v \in V.$$

Let $T = (G, v_1, v_2, v, \alpha)$ be a gate in \mathcal{T} . Suppose $\mathcal{C}(v) = 2k - 1$. Let \mathbf{a}_i^* and \mathbf{l}_i denote the i^{th} row vectors of \mathbf{A}^* and $\mathbf{L}[T]$, respectively; let \mathbf{b}_j^* and \mathbf{r}_j denote the j^{th} column vectors of \mathbf{B}^* and $\mathbf{R}[T]$, respectively.

From Property 1, $\mathbf{L}[T]$ and $\mathbf{R}[T]$ are the only two gadget matrices that modify the entries in rows $\mathbf{a}_{2k-1}^*, \mathbf{a}_{2k}^*$ and in columns $\mathbf{b}_{2k-1}^*, \mathbf{b}_{2k}^*$, in the transformation from the prototype \mathcal{G}^* to $\mathcal{G}^{\mathcal{S}}$. Thus, we have

$$\mathbf{a}_{2k-1}^{\mathcal{S}} = \mathbf{a}_{2k-1}^{*} + \mathbf{l}_{2k-1}, \quad \mathbf{a}_{2k}^{\mathcal{S}} = \mathbf{a}_{2k}^{*} + \mathbf{l}_{2k}; \quad \text{and}$$
(3)

$$\mathbf{b}_{2k-1}^{\mathcal{S}} = \mathbf{b}_{2k-1}^{*} + \mathbf{r}_{2k-1}, \quad \mathbf{b}_{2k}^{\mathcal{S}} = \mathbf{b}_{2k}^{*} + \mathbf{r}_{2k}.$$
 (4)

Now, we prove $\overline{\mathbf{x}}$ satisfies constraint $\mathcal{P}[T, \epsilon]$. Here we only consider the case when $G = G_+$. In this case, we need to prove $\overline{\mathbf{x}}[v] = \min(\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2], 1/K) \pm \epsilon$. Proofs for other types of gates are similar and can be found in **Appendix D**.

Since $\mathbf{a}_{2k-1}^* = \mathbf{a}_{2k}^*$ and $\mathbf{b}_{2k-1}^* = \mathbf{b}_{2k}^*$, from (3), (4) and Figure 3, we have

$$\left\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \right\rangle - \left\langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \right\rangle = \overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] - \overline{\mathbf{x}}[v], \text{ and}$$
(5)

$$\left\langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \right\rangle - \left\langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \right\rangle = \overline{\mathbf{y}}[v] - \left(\overline{\mathbf{y}}_{C}[v] - \overline{\mathbf{y}}[v] \right).$$
(6)

In a proof by contradiction, we consider two cases. First, we assume $\overline{\mathbf{x}}[v] > \min(\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2], 1/K) + \epsilon$. Since $\overline{\mathbf{x}}[v] \le 1/K + \epsilon$, the assumption would imply $\overline{\mathbf{x}}[v] > \overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] + \epsilon$. By Equation (5), we have $\overline{\mathbf{y}}[v] = y_{2k-1} = 0$, because (\mathbf{x}, \mathbf{y}) is an ϵ -well-supported Nash equilibrium. On the other hand, since $\overline{\mathbf{y}}_C[v] = 1/K \pm \epsilon \gg \epsilon$, by Equation (6), we have $\overline{\mathbf{x}}[v] = x_{2k-1} = 0$, contradicting our assumption that $\overline{\mathbf{x}}[v] > \overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] + \epsilon > 0$.

Next, we assume $\overline{\mathbf{x}}[v] < \min(\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2], 1/K) - \epsilon \leq \overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] - \epsilon$. Then, Equation (5) implies $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$. By Equation (6), we have $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v]$ and thus, $\overline{\mathbf{x}}[v] \geq 1/K - \epsilon$, which contradicts our assumption that $\overline{\mathbf{x}}[v] < \min(\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2], 1/K) - \epsilon \leq 1/K - \epsilon$.

We have now completed the proof of Lemma 5.7. To prove our main technical Theorem 5.1, we only need to prove Theorem 4.6 and Lemma 5.6.

7 **PPAD-Completeness of** BROUWER^{*f*}

To prove Theorem 4.6, we reduce a two-dimensional instance of BROUWER^{f_2}, that is, a valid 3coloring of a 2-dimensional grid, to BROUWER^f, where recall $l f_2(n) = \lfloor n/2 \rfloor$. The basic idea of the reduction is to iteratively embed an instance of BROUWER into a hypergrid one dimension higher to eventually "fold" or embed this two-dimensional input instance into the desired hypergrid. We use the following concept to describe our embedding processes. A triple $T = (C, d, \mathbf{r})$ is a *coloring triple* if $\mathbf{r} \in \mathbb{Z}^d$ with $r_i \geq 7$ for all $1 \leq i \leq d$ and C is a valid Brouwer-mapping circuit with parameters d and \mathbf{r} . Let Size [C] denote the number of gates plus the number of input and output variables in a circuit C.

Our embedding is carried out by a sequence of three polynomial-time transformations: $\mathbf{L}^1(T, t, u)$, $\mathbf{L}^2(T, u)$, and $\mathbf{L}^3(T, t, a, b)$. They embed a coloring triple T into a larger T' (that is, the volume of the search space of T' is greater than the one of T) such that from every panchromatic simplex of T', one can find a panchromatic simplex of T efficiently.

To simplify our proof, in the context of this section, we slightly modify the definition of BROUWER^f: In the original definition, each valid Brouwer-mapping circuit C defines a color assignment from the search space to $\{1, 2, 3, ..., d, d + 1\}$. In this section, we replace the color d + 1 by a special color "red". In other words, if the output bits of C evaluated at \mathbf{p} satisfy Case i with $1 \leq i \leq d$, then $\operatorname{Color}_C[\mathbf{p}] = i$; otherwise, the output bits satisfy Case d + 1, and $\operatorname{Color}_C[\mathbf{p}] =$ "red".

We first prove a useful property of valid Brouwer-mapping circuits.

Property 2 (Boundary Continuity). Let C be a valid Brouwer-mapping circuit with parameters d and **r**. If points $\mathbf{p}, \mathbf{p}' \in \partial(A_{\mathbf{r}}^d)$ satisfy $\mathbf{p}' = \mathbf{p} + \mathbf{e}_t$ for some $1 \le t \le d$ and $1 \le p_t \le r_t - 2$, then $Color_C[\mathbf{p}] = Color_C[\mathbf{p}']$.

Proof. By the definition, if C is a valid Brouwer-mapping circuit C with parameters d and \mathbf{r} , then for each $\mathbf{p} \in \partial (A_{\mathbf{r}}^d)$, $\operatorname{Color}_C [\mathbf{p}]$ has the following property: If there exists an $i \in [1:d]$ such that $p_i = 0$, then $\operatorname{Color}_C [\mathbf{p}] = \max\{i \mid p_i = 0\}$; otherwise, $\forall i, p_i \neq 0$ and $\exists i, p_i = r_i - 1$, we have $\operatorname{Color}_C [\mathbf{p}] =$ "red". Thus, if $1 \leq p_t \leq r_t - 2$, then $\operatorname{Color}_C [\mathbf{p}] = \operatorname{Color}_C [\mathbf{p}']$.

7.1 Reductions Among Coloring Triples

Both $\mathbf{L}^1(T, t, u)$ and $\mathbf{L}^2(T, u)$ are very simple operations:

 $\operatorname{\mathbf{Color}}_{C'}[\mathbf{p}]$ of a point $\mathbf{p} \in A^d_{\mathbf{r}'}$ assigned by $(C', d, \mathbf{r}') = \mathbf{L}^1(T, t, u)$

```
1: if \mathbf{p} \in \partial \left(A_{\mathbf{r}'}^d\right) then

2: if there exists i such that p_i = 0 then

3: \operatorname{Color}_{C'} \left[\mathbf{p}\right] = i_{\max} = \max\{i \mid p_i = 0\}

4: else

5: \operatorname{Color}_{C'} \left[\mathbf{p}\right] = \operatorname{red}

6: else if p_t \leq r_t then

7: \operatorname{Color}_{C'} \left[\mathbf{p}\right] = \operatorname{Color}_C \left[\mathbf{p}\right]

8: else

9: \operatorname{Color}_{C'} \left[\mathbf{p}\right] = \operatorname{red}
```

Figure 4: How $\mathbf{L}^1(T, t, u)$ extends the coloring triple $T = (C, d, \mathbf{r})$

- Given a coloring triple $T = (C, d, \mathbf{r})$ and two integers $1 \le t \le d, u > r_t, \mathbf{L}^1(T, t, u)$ pads dimension t to size u, i.e., it builds a new coloring triple $T' = (C', d, \mathbf{r}')$ with $r'_t = u$ and $r'_i = r_i$, for all $i : 1 \le i \ne t \le d$.
- For integer $u \ge 7$, $\mathbf{L}^2(T, u)$ adds a dimension to T by constructing $T' = (C', d+1, \mathbf{r}')$ such that $\mathbf{r}' \in \mathbb{Z}^{d+1}$, $r'_{d+1} = u$ and $r'_i = r_i$, for all $i \in [1; d]$.

These two transformations are described in Figure 4 and Figure 5, respectively. We prove their properties in the following two lemmas.

Lemma 7.1 ($\mathbf{L}^1(T, t, u)$): Padding a Dimension). Given a coloring triple $T = (C, d, \mathbf{r})$ and two integers $1 \le t \le d$ and $u > r_t$, we can construct a new coloring triple $T' = (C', d, \mathbf{r}')$ that satisfies the following two conditions:

- **A.** For all $i : 1 \le i \ne t \le d$, $r'_i = r_i$, and $r'_t = u$. In addition, there exists a polynomial $g_1(n)$ such that Size $[C'] = \text{Size } [C] + O(g_1(\text{Size } [\mathbf{r}']))$ and T' can be computed in time polynomial in Size [C']. We write $T' = \mathbf{L}^1(T, t, u)$;
- **B.** From each panchromatic simplex P' of coloring triple T', we can compute a panchromatic simplex P of T in polynomial time.

Proof. We define circuit C' by its color assignment in Figure 4. Property **A** is true according to this definition.

To show Property **B**, let $P' \subset K_{\mathbf{p}}$ be a panchromatic simplex of T'. We first note that $p_t < r_t - 1$, because had $p_t \ge r_t - 1$, $K_{\mathbf{p}}$ would not contain color t according to the color assignment. Thus, it follows from $\operatorname{Color}_{C'}[\mathbf{q}] = \operatorname{Color}_C[\mathbf{q}]$ for each $\mathbf{q} \in A_{\mathbf{r}}^d$ that P' is also a panchromatic simplex of the coloring triple T.

 $\operatorname{Color}_{C'}[\mathbf{p}]$ of a point $\mathbf{p} \in A^{d+1}_{\mathbf{r}'}$ assigned by $(C', d+1, \mathbf{r}') = \mathbf{L}^2(T, u)$

1: if $\mathbf{p} \in \partial \left(A_{\mathbf{r}'}^d\right)$ then 2: if there exists *i* such that $p_i = 0$ then 3: $\operatorname{Color}_{C'} [\mathbf{p}] = i_{\max} = \max\{i \mid p_i = 0\}$ 4: else 5: $\operatorname{Color}_{C'} [\mathbf{p}] = \operatorname{red}$ 6: else if $p_{d+1} = 1$ then 7: $\operatorname{Color}_{C'} [\mathbf{p}] = \operatorname{Color}_{C} [\hat{\mathbf{p}}]$, where $\hat{\mathbf{p}} \in \mathbb{Z}^d$ satisfying $\hat{p}_i = p_i$ for all $1 \le i \le d$ 8: else 9: $\operatorname{Color}_{C'} [\mathbf{p}] = \operatorname{red}$

Figure 5: How $\mathbf{L}^2(T, u)$ extends the coloring triple $T = (C, d, \mathbf{r})$

Lemma 7.2 ($\mathbf{L}^2(T, u)$: Adding a Dimension). Given a coloring triple $T = (C, d, \mathbf{r})$ and integer $u \geq 7$, we can construct a new coloring triple $T' = (C', d + 1, \mathbf{r}')$ that satisfies the following conditions:

- **A.** For all $i : 1 \leq i \leq d$, $r'_i = r_i$, and $r'_{d+1} = u$. Moreover, there exists a polynomial $g_2(n)$ such that Size $[C'] = \text{Size } [C] + O(g_2(\text{Size } [\mathbf{r}']))$. T' can be computed in time polynomial in Size [C']. We write $T' = \mathbf{L}^2(T, u)$.
- **B.** From each panchromatic simplex P' of coloring triple T', we can compute a panchromatic simplex P of T in polynomial time.

Proof. For each point $\mathbf{p} \in A_{\mathbf{r}'}^{d+1}$, we use $\hat{\mathbf{p}}$ to denote the point $\mathbf{z} \in A_{\mathbf{r}}^d$ with $z_i = p_i, \forall i \in [1:d]$. The color assignment of circuit C' is given in Figure 5. Clearly, Property **A** is true.

To prove Property **B**, we let $P' \subset K_{\mathbf{p}}$ be a panchromatic simplex of T'. We note that $p_{d+1} = 0$, for otherwise, $K_{\mathbf{p}}$ does not contain color d + 1. Note also that $\operatorname{Color}_{C'}[\mathbf{q}] = d + 1$ for every $\mathbf{q} \in A_{\mathbf{r}'}^{d+1}$ with $q_{d+1} = 0$. Thus, for every point $\mathbf{q} \in P'$ with $\operatorname{Color}_{C'}[\mathbf{q}] \neq d + 1$, we have $q_{d+1} = 1$. So, because $\operatorname{Color}_{C'}[\mathbf{q}] = \operatorname{Color}_{C}[\hat{\mathbf{q}}]$ for every $\mathbf{q} \in A_{\mathbf{r}'}^{d+1}$ with $q_{d+1} = 1$, $P = \{\hat{\mathbf{q}} \mid \mathbf{q} \in P' \text{ and } \operatorname{Color}_{C'}[\mathbf{q}] \neq d + 1\}$ is a panchromatic simplex of T.

Transformation $\mathbf{L}^{3}(T, t, a, b)$ is the one that does all the hard work.

Lemma 7.3 ($\mathbf{L}^3(T, t, a, b)$): Snake Embedding). Given a coloring triple $T = (C, d, \mathbf{r})$ and integer $1 \le t \le d$, if $r_t = a(2b+1) + 5$ for two integers $a, b \ge 1$, then we can construct a new triple $T' = (C', d+1, \mathbf{r}')$ that satisfies the following conditions:

A. For $i: 1 \leq i \neq t \leq d$, $r'_i = r_i$ and $r'_t = a + 5$ and $r'_{d+1} = 4b + 3$. Moreover, there exists a polynomial $g_3(n)$ such that Size $[C'] = \text{Size } [C] + O(g_3(\text{Size } [\mathbf{r}']))$ and T' can be computed in time polynomial in Size [C']. We write $T' = \mathbf{L}^3(T, t, a, b)$.



Figure 6: The two dimensional view of set $W \subset A^{d+1}_{\mathbf{r}'}$

B. From each panchromatic simplex P' of coloring triple T', we can compute a panchromatic simplex P of T in polynomial time.

Proof. Consider the domains $A^d_{\mathbf{r}} \subset \mathbb{Z}^d$ and $A^{d+1}_{\mathbf{r}'} \subset \mathbb{Z}^{d+1}$ of our coloring triples. We form the reduction $\mathbf{L}^3(T, t, a, b)$ in three steps. First, we define a *d*-dimensional set $W \subset A^{d+1}_{\mathbf{r}'}$ that is large enough to contain $A^d_{\mathbf{r}}$. Second, we define a map ψ from W to $A^d_{\mathbf{r}}$ that (implicitly) specifies an embedding of $A^d_{\mathbf{r}}$ into W. Finally, we build a circuit C' for $A^{d+1}_{\mathbf{r}'}$ and show that from each panchromatic simplex of C', we can, in polynomial time, compute a panchromatic simplex of C.

A two dimensional view of $W \subset A_{\mathbf{r}'}^{d+1}$ is illustrated in Figure 6. We use a snake-pattern to realize the longer t^{th} dimension of $A_{\mathbf{r}}^d$ in the two-dimensional space defined by the shorter t^{th} and $(d+1)^{th}$ dimensions of $A_{\mathbf{r}'}^{d+1}$. Formally, W consists of points $\mathbf{p} \in A_{\mathbf{r}'}^{d+1}$ satisfying $1 \leq p_{d+1} \leq 4b+1$ and

if
$$p_{d+1} = 1$$
, then $2 \le p_t \le a + 4$;
if $p_{d+1} = 4b + 1$, then $0 \le p_t \le a + 2$;
if $p_{d+1} = 4(b - i) - 1$ where $0 \le i \le b - 1$, then $2 \le p_t \le a + 2$;
if $p_{d+1} = 4(b - i) - 3$ where $0 \le i \le b - 2$, then $2 \le p_t \le a + 2$;
if $p_{d+1} = 4(b - i) - 2$ where $0 \le i \le b - 1$, then $p_t = 2$;
if $p_{d+1} = 4(b - i)$ where $0 \le i \le b - 1$, then $p_t = a + 2$.

To build T', we embed the coloring triple T into W. The embedding is implicitly given by a natural surjective map ψ from W to $A^d_{\mathbf{r}}$, a map that will play a vital role in our construction and analysis. For each $\mathbf{p} \in W$, we use $\mathbf{p}[m]$ to denote the point \mathbf{q} in \mathbb{Z}^d such that $q_t = m$ and $q_i = p_i$, for all $i: 1 \leq i \neq t \leq d$. We define $\psi(\mathbf{p})$ according to the following cases:

if
$$p_{d+1} = 1$$
, then $\psi(\mathbf{p}) = \mathbf{p}[2ab + p_t]$

 $\operatorname{Color}_{C'}[\mathbf{p}]$ of a point $\mathbf{p} \in A^{d+1}_{\mathbf{r}'}$ assigned by $(C', d+1, \mathbf{r}') = \mathbf{L}^3(T, t, a, b)$

```
1: if \mathbf{p} \in W then
  2:
            \operatorname{Color}_{C'}[\mathbf{p}] = \operatorname{Color}_{C}[\psi(\mathbf{p})]
 3: else if \mathbf{p} \in \partial (A_{\mathbf{r}'}^{d+1}) then
            if there exists i such that p_i = 0 then
  4:
                 \operatorname{Color}_{C'}[\mathbf{p}] = i_{\max} = \max\{i \mid p_i = 0\}
  5:
  6:
            else
  7:
                 \operatorname{Color}_{C'}[\mathbf{p}] = \operatorname{red}
  8: else if p_{d+1} = 4i where 1 \le i \le b and 1 \le p_t \le a+1 then
            \operatorname{Color}_{C'}[\mathbf{p}] = d + 1
  9:
10: else if p_{d+1} = 4i + 1, 4i + 2 or 4i + 3 where 0 \le i \le b - 1 and p_t = 1 then
            \operatorname{Color}_{C'}[\mathbf{p}] = d + 1
11:
12: else
13:
            \operatorname{Color}_{C'}[\mathbf{p}] = \operatorname{red}
```

Figure 7: How $\mathbf{L}^{3}(T, t, a, b)$ extends the coloring triple $T = (C, d, \mathbf{r})$

if
$$p_{d+1} = 4b + 1$$
, then $\psi(\mathbf{p}) = \mathbf{p}[p_t]$;
if $p_{d+1} = 4(b-i) - 1$ where $0 \le i \le b - 1$, then $\psi(\mathbf{p}) = \mathbf{p}[(2i+2)a + 4 - p_t]$;
if $p_{d+1} = 4(b-i) - 3$ where $0 \le i \le b - 2$, then $\psi(\mathbf{p}) = \mathbf{p}[(2i+2)a + p_t]$;
if $p_{d+1} = 4(b-i) - 2$ where $0 \le i \le b - 1$, then $\psi(\mathbf{p}) = \mathbf{p}[(2i+2)a + 2]$;
if $p_{d+1} = 4(b-i)$ where $0 \le i \le b - 1$, then $\psi(\mathbf{p}) = \mathbf{p}[(2i+1)a + 2]$.

Essentially, we map W bijectively to $A_{\mathbf{r}}^d$ along its t^{th} dimension with exception that when the snake pattern of W is making a turn, we stop the advance in $A_{\mathbf{r}}^d$, and continue the advance after it completes the turn. Let $\psi_i(\mathbf{p})$ denote the i^{th} component of $\psi(\mathbf{p})$. Our embedding scheme guarantees the following important property of ψ .

Property 3 (Boundary Preserving). Let \mathbf{p} be a point in $W \cap \partial(A_{\mathbf{r}'}^{d+1})$. If there exists i such that $p_i = 0$, then $\max\{i \mid p_i = 0\} = \max\{i \mid \psi_i(\mathbf{p}) = 0\}$. Otherwise, all entries of \mathbf{p} are non-zero and there exists l such that $p_l = r'_l - 1$, in which case, all entries of point $\psi(\mathbf{p})$ are nonzero and $\psi_l(\mathbf{p}) = r_l - 1$.

The circuit C' specifies a color assignment of $A_{\mathbf{r}'}^{d+1}$ according to Figure 7. C' is derived from circuit C and map ψ . By Property 3, we can verify that C' is a valid Brouwer-mapping circuit with parameters d + 1 and \mathbf{r}' .

Property \mathbf{A} follows directly from our construction. In order to establish Property \mathbf{B} of the lemma, we prove the following collection of statements to cover all possible cases of the given

panchromatic simplex P' of T'. In the following statements, P' is a panchromatic simplex of T'in $A_{\mathbf{r}'}^{d+1}$ and let $\mathbf{p}^* \in A_{\mathbf{r}'}^{d+1}$ be the point such that $P' \subset K_{\mathbf{p}^*}$. We will also use the following notation: For each $\mathbf{p} \in A_{\mathbf{r}'}^{d+1}$, we will use $\mathbf{p}[m_1, m_2]$ to denote the point $\mathbf{q} \subset \mathbb{Z}^{d+1}$ such that $q_t = m_1, q_{d+1} = m_2$ and $q_i = p_i$ for all $i: 1 \leq i \neq t \leq d$.

Statement 1. If $p_t^* = 0$, then $p_{d+1}^* = 4b$ and furthermore, for every point $\mathbf{p} \in P'$ such that $Color_{C'}[\mathbf{p}] \neq d+1$, $Color_C[\psi(\mathbf{p}[p_t, 4b+1])] = Color_{C'}[\mathbf{p}]$.

Proof. First, note that $p_{d+1}^* \neq 4b + 1$, for otherwise, $K_{\mathbf{p}^*}$ does not contain color d + 1. Second, if $p_{d+1}^* < 4b$, then each point $\mathbf{q} \in K_{\mathbf{p}^*}$ is colored according one of the conditions in line 3, 8 or 10 of Figure 7. Let $\mathbf{q}^* \in K_{\mathbf{p}^*}$ be the "red" point in P'. Then, \mathbf{q}^* must satisfy the condition in line 6 and hence there exists l such that $q_l^* = r_l' - 1$. By our assumption, $p_t^* = 0$. Thus, if $p_{d+1}^* < 4b$, then $l \notin \{t, d+1\}$, implying for each $\mathbf{q} \in K_{\mathbf{p}^*}$, $q_l > 0$ (as $q_l \ge q_l^* - 1 > 0$) and $\operatorname{Color}_{C'}[\mathbf{q}] \ne l$. Then, $K_{\mathbf{p}^*}$ does not contain color l, contradicting the assumption of the statement. Putting these two cases together, we have $p_{d+1}^* = 4b$.

We now prove the second part of the statement. If $p_{d+1} = 4b + 1$, then we are done, because $\operatorname{Color}_{C}[\psi(\mathbf{p})] = \operatorname{Color}_{C'}[\mathbf{p}]$ according to line 1 of Figure 7. Let us assume $p_{d+1} = 4b$. Since the statement assumes $\operatorname{Color}_{C'}[\mathbf{p}] \neq d+1$, \mathbf{p} satisfies the condition in line 3 and hence $\mathbf{p} \in \partial(A_{\mathbf{r}'}^{d+1})$. By Property 2, we have $\operatorname{Color}_{C'}[\mathbf{p}[p_t, 4b+1]] = \operatorname{Color}_{C'}[\mathbf{p}]$, completing the proof of the statement.

Statement 2. If $p_t^* = a + 2$ or a + 3, then $p_{d+1}^* = 0$, and in addition, for each $\mathbf{p} \in P'$ such that $Color_{C'}[\mathbf{p}] \neq d+1$, $Color_C[\psi(\mathbf{p}[p_t, 1])] = Color_{C'}[\mathbf{p}]$.

Proof. If $p_{d+1}^* > 0$, then $K_{\mathbf{p}^*}$ does not contain color d + 1. So $p_{d+1}^* = 0$. In this case, p_{d+1} must be 1, since $\operatorname{Color}_{C'}[\mathbf{q}] = d + 1$ for any $\mathbf{q} \in A_{\mathbf{r}'}^{d+1}$ with $q_{d+1} = 0$. Thus, $\operatorname{Color}_{C}[\psi(\mathbf{p}[p_t, 1])] = \operatorname{Color}_{C'}[\mathbf{p}[p_t, 1]] = \operatorname{Color}_{C'}[\mathbf{p}]$.

Statement 3. If $p_{d+1}^* = 4b$, then $0 \le p_t^* \le a+1$. Moreover, for each $\mathbf{p} \in P'$ such that $Color_{C'}[\mathbf{p}] \ne d+1$, $Color_C[\psi(\mathbf{p}[p_t, 4b+1])] = Color_{C'}[\mathbf{p}]$.

Proof. The first part of the statement is straightforward. Similar to the proof of Statement 1, we can prove the second part for the case when $0 \le p_t \le a + 1$. When $p_t = a + 2$, we have $\psi(\mathbf{p}) = \psi(\mathbf{p}[p_t, 4b + 1])$. Thus, $\operatorname{Color}_C[\psi(\mathbf{p}[p_t, 4b + 1])] = \operatorname{Color}_C[\psi(\mathbf{p})] = \operatorname{Color}_{C'}[\mathbf{p}]$.

We can similarly prove the following statements.

Statement 4. If $p_{d+1}^* = 4i + 1$ or 4i + 2 for some $0 \le i \le b - 1$, then $p_t^* = 1$. Moreover, for each $\mathbf{p} \in P'$ such that $Color_{C'}[\mathbf{p}] \ne d + 1$, $Color_C[\psi(\mathbf{p}[2, p_{d+1}])] = Color_{C'}[\mathbf{p}]$.

Statement 5. If $p_{d+1}^* = 4i$ for some $1 \le i \le b-1$, then $1 \le p_t^* \le a+1$. In addition, for each $\mathbf{p} \in P'$ such that $Color_{C'}[\mathbf{p}] \ne d+1$, if $2 \le p_t \le a+1$, then $Color_C[\psi(\mathbf{p}[p_t, 4i+1])] = Color_{C'}[\mathbf{p}]$; if $p_t = 1$, then $Color_C[\psi(\mathbf{p}[2, 4i+1])] = Color_{C'}[\mathbf{p}]$.

Statement 6. If $p_{d+1}^* = 4i - 1$ for some $1 \le i \le b$, then $1 \le p_t^* \le a + 1$. Moreover, for each $\mathbf{p} \in P'$ such that $Color_{C'}[\mathbf{p}] \ne d+1$, if $2 \le p_t \le a+1$, then $Color_C[\psi(\mathbf{p}[p_t, 4i - 1])] = Color_{C'}[\mathbf{p}]$; if $p_t = 1$, then $Color_C[\psi(\mathbf{p}[2, 4i - 1])] = Color_{C'}[\mathbf{p}]$.

Statement 7. If $p_{d+1}^* = 0$, then $1 \le p_t^* \le a+3$. In addition, for each $\mathbf{p} \in P'$ such that $Color_{C'}[\mathbf{p}] \ne d+1$, if $2 \le p_t^* \le a+1$, then $Color_C[\psi(\mathbf{p}[p_t, 1])] = Color_{C'}[\mathbf{p}]$; if $p_t^* = 1$, then $Color_C[\psi(\mathbf{p}[2, 1])] = Color_{C'}[\mathbf{p}]$.

In addition,

Statement 8. $p_{d+1}^* \neq 4b + 1$.

Proof. If $p_{d+1}^* = 4b + 1$ then $K_{\mathbf{p}^*}$ does not contain color d + 1.

Now suppose that P' is a panchromatic simplex of T'. Let $\mathbf{p}^* \in \mathbb{Z}^{d+1}$ be the point such that $P' \subset K_{\mathbf{p}^*}$. Then, P' and \mathbf{p}^* must satisfy the conditions of one of the statements above. By that statement, we can transform every point $\mathbf{p} \in P'$, (aside from the one that has color d + 1) back to a point \mathbf{q} in $A^d_{\mathbf{r}}$ to obtain a set P from P'. Since P is accommodated, it is a panchromatic simplex of C. Thus, with all the statements above, we specify an efficient algorithm to compute a panchromatic simplex P of T given a panchromatic simplex P' of T'.

7.2 PPAD-Completeness of Problem BROUWER^f

We are now ready to prove the main result of this section.

Proof of Theorem 4.6. We reduce BROUWER^{f_2} to BROUWER^f in order to prove that the latter is **PPAD**-complete. Recall, $f_2(n) = \lfloor n/2 \rfloor$. Suppose $(C, 0^{2n})$ is an input instance of BROUWER^{f_2}. Let

$$l = f(11n) \ge 3$$
, $m' = \left\lceil \frac{n}{l-2} \right\rceil$ and $m = \left\lceil \frac{11n}{l} \right\rceil$.

We iteratively construct a sequence of coloring triples $\mathcal{T} = \{T^0, T^1, \dots, T^{w-1}, T^w\}$ for some w = O(m), starting with $T^0 = (C, 2, (2^n, 2^n))$ and ending with $T^w = (C^w, m, \mathbf{r}^w)$ where $\mathbf{r}^w \in \mathbb{Z}^m$ and $r_i^w = 2^l$, for all $i \in [1 : m]$. At the t^{th} iteration, we apply either $\mathbf{L}^1, \mathbf{L}^2$ or \mathbf{L}^3 with properly chosen parameters to build T^{t+1} from T^t .

Below we give the details of our construction. In the first step, we call $\mathbf{L}^{1}(T^{0}, 1, 2^{m'(l-2)})$ to get $T^{1} = (C^{1}, 2, (2^{m'(l-2)}, 2^{n}))$. This step is possible because $m'(l-2) \geq n$. We then invoke the procedure in Figure 8. In each for-loop, the first component of \mathbf{r} decreases by a factor of 2^{l-2} , while the dimension of the space increases by 1. After running the for-loop (m'-5) times, we obtain a coloring triple $T^{3m'-14} = (C^{3m'-14}, d^{3m'-14}, \mathbf{r}^{3m'-14})$ that satisfies⁴

$$d^{3m'-14} = m'-3$$
, $r_1^{3m'-14} = 2^{5(l-2)}$, $r_2^{3m'-14} = 2^n$ and $r_i^{3m'-14} = 2^l$, $\forall i \in [3:m'-3]$.

⁴ Remark: the superscript of C, d, r_i , denotes the index of the iterative step. It is not an exponent!

The Construction of $T^{3m'-14}$ from T^1

1: for any t from 0 to m' - 6 do 2: It can be proved inductively that $T^{3t+1} = (C^{3t+1}, d^{3t+1}, \mathbf{r}^{3t+1})$ satisfies $d^{3t+1} = t + 2, r_1^{3t+1} = 2^{(m'-t)(l-2)}, r_2^{3t+1} = 2^n$ and $r_i^{3t+1} = 2^l$ for all $3 \le i \le t + 2$ 3: let $u = (2^{(m'-t-1)(l-2)} - 5)(2^{l-1} - 1) + 5$ 4: $[u \ge r_1^{3t+1} = 2^{(m'-t)(l-2)}$ under the assumption that $t \le m' - 6$ and $l \ge 3$] 5: $T^{3t+2} = \mathbf{L}^1(T^{3t+1}, 1, u)$ 6: $T^{3t+3} = \mathbf{L}^3(T^{3t+2}, 1, 2^{(m'-t-1)(l-2)}, 2^{l-2} - 1)$ 7: $T^{3t+4} = \mathbf{L}^1(T^{3t+3}, t+3, 2^l)$

Figure 8: The Construction of $T^{3m'-14}$ from T^1

The Construction of $T^{w'}$ from $T^{3m'-14}$

1: let t = 02: while $T^{3(m'+t)-14} = (C^{3(m'+t)-14}, m'+t-3, \mathbf{r}^{3(m'+t)-14})$ satisfies $r_1^{3(m'+t)-14} > 2^l$ do 3: let $k = \lceil (r_1^{3(m'+t)-14} - 5)/(2^{l-1} - 1) \rceil + 5$ 4: $T^{3(m'+t)-13} = \mathbf{L}^1 (T^{3(m'+t)-14}, 1, (k-5)(2^{l-1} - 1) + 5)$ 5: $T^{3(m'+t)-12} = \mathbf{L}^3 (T^{3(m'+t)-13}, 1, k, 2^{l-2} - 1)$ 6: $T^{3(m'+t)-11} = \mathbf{L}^1 (T^{3(m'+t)-12}, m'+t-2, 2^l)$, set t = t + 17: let w' = 3(m'+t) - 13 and $T^{w'} = \mathbf{L}^1 (T^{3(m'+t)-14}, 1, 2^l)$

Figure 9: The Construction of $T^{w'}$ from $T^{3m'-14}$

Next, we call the procedure given in Figure 9. Note that the while-loop must terminate in at most 8 iterations because we start with $r_1^{3m'-14} = 2^{5(l-2)}$. The procedure returns a coloring triple $T^{w'} = (C^{w'}, d^{w'}, \mathbf{r}^{w'})$ that satisfies

$$w' \leq 3m' + 11, \ d^{w'} \leq m' + 5, \ r_1^{w'} = 2^l, \ r_2^{w'} = 2^n \ \text{ and } \ r_i^{w'} = 2^l, \ \forall \ i \in [3:d^{w'}].$$

We then repeat the whole process above on the second coordinate and obtain a coloring triple $T^{w''} = (C^{w''}, d^{w''}, \mathbf{r}^{w''})$ that satisfies

$$w'' \le 6m' + 21, \ d^{w''} \le 2m' + 8 \ \text{ and } \ r_i^{w''} = 2^l, \ \forall \ i \in [1 \ d^{w''}].$$

The way in which we define m and m' guarantees

$$d^{w''} \le 2m' + 8 \le 2\left(\frac{n}{l-2} + 1\right) + 8 \le 2\left(\frac{n}{l/3}\right) + 10 = \frac{6n}{l} + 10 \le \frac{11n}{l} \le m.$$

Finally, by applying \mathbf{L}^2 on coloring triple $T^{w''}$ for $m - d^{w''}$ times with parameter $u = 2^l$, we obtain $T^w = (C^w, m, \mathbf{r}^w)$ with $\mathbf{r}^w \in \mathbb{Z}^m$ and $r_i^w = 2^l$, $\forall i \in [1 : m]$. It follows from our construction that w = O(m).

To see why the sequence \mathcal{T} gives a reduction from BROUWER^{f_2} to BROUWER^{f_2} to BROUWER^{f_2}, let $T^i = (C^i, d^i, \mathbf{r}^i)$ (again the superscript of C, d, \mathbf{r} , denotes the index of the iteration). As sequence $\{ \text{Size } [\mathbf{r}^i] \}_{0 \le i \le w}$ is nondecreasing and w = O(m) = O(n), by the Property A of Lemma 7.1, 7.2 and 7.3, there exists a polynomial g(n) such that

Size
$$[C^w]$$
 = Size $[C] + O(g(n))$.

By these Properties **A** again, we can construct the whole sequence \mathcal{T} and in particular, triple $T^w = (C^w, m, \mathbf{r}^w)$, in time polynomial in Size [C].

Pair $(C^w, 0^{11n})$ is an input instance of BROUWER^f. Given any panchromatic simplex P of $(C^w, 0^{11n})$ and using the algorithms in Properties **B** of Lemma 7.1, 7.2 and 7.3, we can compute a sequence of panchromatic simplices $P^w = P, P^{w-1}..., P^0$ iteratively in polynomial time, where P^t is a panchromatic simplex of T^t and is computed from the panchromatic simplex P^{t+1} of T^{t+1} . In the end, we obtain P^0 , which is a panchromatic simplex of $(C, 0^{2n})$.

8 Computing Fixed Points with Generalized Circuits

In this section, we show that fixed points can be modeled by generalized circuits. In particular, we reduce the search of a panchromatic simplex in an instance of BROUWER^{f_1} (which will be simply referred to as BROUWER in this section) to POLY³-GCIRCUIT, the computation of a $1/K^3$ -approximate solution of a generalized circuit of K nodes. Recall $f_1(n) = 3$. Our reduction will use ideas from [18]. However, we will need to develop several new techniques to meet the geometric and combinatorial challenges in the consideration of high-dimensional fixed points.

Suppose $U = (C, 0^{3n})$ is an input instance of BROUWER, which colors the hypergrid $B^n = \mathbb{Z}_{[0,7]}^n$ with colors from $\{1, ..., n, n+1\}$. Let m be the smallest integer such that $2^m \geq \text{Size}[C] > n$ and $K = 2^{6m}$, where Size[C] is the number of gates plus the number of input and output variables in a Boolean circuit C. Please note that $m = O(\log |C|)$ and hence $2^{\Theta(m)}$ is polynomial in the input size of U.

We will construct a generalized circuit $S^U = (V, T^U)$ with |V| = K in polynomial time. Our construction ensures that,

• Property **R**: From every $(1/K^3)$ -approximate solution to \mathcal{S}^U , we can compute a panchromatic simplex P of circuit C in polynomial time.

In the rest of this section, we assume $\epsilon = 1/K^3$.

8.1 Overcome the Curse of Dimensionality

We prove a key geometric lemma (Lemma 8.2 below) for finding a panchromatic simplex in order to overcome the curse of dimensionality. Our construction of S^U will then build on this lemma. For $a \in \mathbb{R}^+$, let $\pi(a) = \max\{i \mid 0 \le i \le 7 \text{ and } i < a\}$ be the largest integer in [0:7] that is smaller than a. Let $E^n = \{\mathbf{z}^1, \mathbf{z}^2, ..., \mathbf{z}^n, \mathbf{z}^{n+1}\}$ where $\mathbf{z}^i = \mathbf{e}_i/K^2$ and $\mathbf{z}^{n+1} = -\sum_{1 \le i \le n} \mathbf{e}_i/K^2$. For each $i \in [1:n+1]$, we encode the i^{th} color in Color_C by vector \mathbf{z}^i .

For any $\mathbf{p} \in \mathbb{R}^n_+$, let $\mathbf{q} = \pi(\mathbf{p})$ be the integer point in $B^n = \mathbb{Z}^n_{[0,7]}$ with $q_i = \pi(p_i)$. Let $\xi(\mathbf{p}) = \mathbf{z}^t$, where $t = \text{Color}_C[\pi(\mathbf{p})]$.

Definition 8.1 (Well-Positioned Points). A real number $a \in \mathbb{R}^+$ is poorly-positioned if there is an integer $t \in [0:7]$ such that $|a - t| \leq 80K\epsilon = 80/K^2$. A point $\mathbf{p} \in \mathbb{R}^n_+$ is well-positioned if none of its components is poorly-positioned, otherwise, it is poorly-positioned.

Let $S = \{\mathbf{p}^1, \mathbf{p}^2, ..., \mathbf{p}^h\}$ be a set of h points in $\mathbb{R}^n_{[0,8]}$. We define

$$I_B(S) = \{ k \mid \mathbf{p}^k \text{ is poorly-positioned } \} \text{ and } I_G(S) = \{ k \mid \mathbf{p}^k \text{ is well-positioned } \}.$$

Lemma 8.2 (Key Geometry: Equiangle Averaging). Suppose $U = (C, 0^{3n})$ is an instance of BROUWER. Let $S = \{\mathbf{p}^i, 1 \le i \le n^3\}$ be n^3 points in $\mathbb{R}^n_{[0,8]}$ such that $\mathbf{p}^i = \mathbf{p}^{i-1} + \sum_{i=1}^n \mathbf{e}_i/K$. If there is a vector $\mathbf{r}^k \in \mathbb{R}^n_{[0,1/K^2]}$ for each k in $I_B(S)$, such that,

$$\left\| \sum_{k \in I_G(S)} \xi(\mathbf{p}^k) + \sum_{k \in I_B(S)} \mathbf{r}^k \right\|_{\infty} = O(\epsilon),$$

then $Q = \{\pi(\mathbf{p}^k), k \in I_G(S)\}$ is a panchromatic simplex of C.

Proof. We first prove that $Q' = \{ \mathbf{q}^k = \pi(\mathbf{p}^k), 1 \leq k \leq n^3 \}$ is accommodated, and satisfies $|Q'| \leq n+1$. Let $\mathbf{q}^k = \pi(\mathbf{p}^k)$ for each $k \in [1:n^3]$. As sequence $\{\mathbf{p}^k\}_{1 \leq k \leq n^3}$ is strictly increasing, $\{\mathbf{q}^k\}_{1 \leq k \leq n^3}$ is non-decreasing. Since $n/K \ll 1$, there exists at most one k_i for each $i \in [1:n]$, such that $q_i^{k_i} = q_i^{k_i-1} + 1$, which implies that Q' is accommodated. Since $\{\mathbf{q}^k\}$ is non-decreasing, $|Q'| \leq n+1$. Because $Q \subset Q'$, Q is accommodated and $|Q| \leq n+1$.

Next, we give an upper bound for $|I_B(S)|$. Because $1/K^2 \ll 1/K \ll 1$, there is at most one k_i for each *i*, such that $p_i^{k_i}$ is poorly-positioned. Since every poorly-positioned point has at least one poorly-positioned component, $|I_B(S)| \le n$ and $|I_G(S)| \ge n^3 - n$.

Let W_i denote the number of points in $\{\mathbf{q}^k : k \in I_G(S)\}$ that are colored *i* by circuit *C*. To prove *Q* is a panchromatic simplex, it suffices to show that $W_i > 0$ for all $i \in [1 : n + 1]$.

Let $\mathbf{r}^G = \sum_{k \in I_G(S)} \xi(\mathbf{p}^k)$ and $\mathbf{r}^B = \sum_{k \in I_B(S)} \mathbf{r}^k$. Since $|I_B(S)| \le n$ and $\|\mathbf{r}^k\|_{\infty} \le 1/K^2$,

$$\|\mathbf{r}^{B}\|_{\infty} \leq n/K^{2}, \text{ and} \\ \|\mathbf{r}^{G}\|_{\infty} \leq \|\mathbf{r}^{B}\|_{\infty} + O(\epsilon) \leq n/K^{2} + O(\epsilon).$$
(7)

Assume by way of contradiction that one of W_i is zero:

- If $W_{n+1} = 0$, supposing $W_{i^*} = \max_{1 \le i \le n} W_i$, then $W_{i^*} \ge n^2 1$, as $|I_G(S)| \ge n^3 n$. But $r_{i^*}^G \ge (n^2 1)/K^2 \gg n/K^2 + O(\epsilon)$, which contradicts (7) above, since $\epsilon = 1/K^3$.
- If $W_t = 0$ for $t \in [1:n]$, then we can assert $W_{n+1} \le n^2/2$, for otherwise, $|r_t^G| > n^2/(2K^2) \gg 1$

EXTRACTBITS $(\mathcal{S}, v, v^1, v^2, v^3)$

- 1: pick unused nodes $v_1, v_2, v_3, v_4 \in V$
- 2: INSERT $(\mathcal{S}, (G_{=}, v, nil, v_1, nil))$
- 3: **for** *j* from 1 to 3 **do**
- 4: pick unused $v_{j1}, v_{j2} \in V$
- 5: INSERT $(\mathcal{S}, (G_{\zeta}, nil, nil, v_{j1}, 2^{-(6m+j)}))$, INSERT $(\mathcal{S}, (G_{<}, v_{j1}, v_{j}, v^{j}, nil))$
- 6: INSERT $(\mathcal{S}, (G_{\times\zeta}, v^j, nil, v_{j2}, 2^{-j}))$, INSERT $(\mathcal{S}, (G_-, v_j, v_{j2}, v_{j+1}, nil))$

Figure 10: Function EXTRACTBITS

 $n/K^2 + O(\epsilon)$, contradicting (7). Suppose $W_{i^*} = \max_{1 \le i \le n+1} W_i$. Then, $W_{i^*} \ge n^2 - 1$ and $i^* \ne n+1$. So $r_{i^*}^G \ge (n^2 - 1 - n/2)/K^2 \gg n/K^2 + O(\epsilon)$, contradicting (7).

As a result, $W_i > 0$ for all $i \in [1: n+1]$, and we have completed the proof of the lemma.

8.2 Construction of the Generalized Circuit S^{U}

We will show how to implement Lemma 8.2, using a generalized circuit. Given an input $U = (C, 0^{3n})$ of BROUWER, our objective is to design a generalized circuit $S^U = (V, \mathcal{T}^U)$ with |V| = K, such that, from any ϵ -approximate solution to S^U , one can find a panchromatic simplex of C in polynomial time. Recall that $\epsilon = 1/K^3$.

More precisely, we will design the generalized circuit \mathcal{S}^U that encodes n^3 points in $\mathbb{R}^n_{[0,8]}$, simulates the π function, and simulates the boolean circuit C. Our \mathcal{S}^U has the property that in each of its ϵ -approximate solution, the sum of the n^3 vectors as given in Lemma 8.2 is close to zero, i.e., $O(\epsilon)$. Then Q, as defined in Lemma 8.2, is a panchromatic simplex of C, which can be computed from the approximate solution of \mathcal{S}^U in polynomial time.

Let us define some notations that will be useful. Suppose $S = (V, \mathcal{T})$ is a generalized circuit with |V| = K. A node $v \in V$ is said to be *unused* in S if none of the gates $T \in \mathcal{T}$ uses v as its output node. Now, suppose $T \notin \mathcal{T}$ is a gate such that the output node of T is unused in S. We will use INSERT(S, T) to denote the insertion of T into S. After calling INSERT(S, T), S becomes $(V, \mathcal{T} \cup \{T\})$.

To encode these n^3 points, let $\{v_i^k\}_{1 \le k \le n^3, 1 \le i \le n}$ be n^4 distinguished nodes in V. We start with $\mathcal{S}^U = (V, \emptyset)$ and insert a number of gates into it so that, in every ϵ -approximate solution \mathbf{x} , values of these nodes encode n^3 points $S = \{\mathbf{p}^k : 1 \le k \le n^3\}$ in $\mathbb{R}^n_{[0,8]}$ that approximately satisfy all the conditions of Lemma 8.2. In our encoding, we let $p_i^k = 8K\mathbf{x}[v_i^k]$.

We define two functions EXTRACTBITS and COLORINGSIMULATION. They are the building blocks in our construction. EXTRACTBITS implements the π function, and is given in Figure 10.

Lemma 8.3 (Encoding Binary). Suppose S = (V, T) is a generalized circuit with |V| = K. For each $v \in V$ and three unused nodes $v^1, v^2, v^3 \in V$, we let S' be the generalized circuit obtained after calling EXTRACTBITS(S, v, v^1, v^2, v^3). Then, in every ϵ -approximate solution \mathbf{x} of S', if $a = 8K\mathbf{x}[v]$ is well-positioned, then $\mathbf{x}[v^i] = {\epsilon \atop B} b_i$, where $b_1b_2b_3$ is the binary representation of integer $\pi(a) \in [0:7]$.

Proof. First we consider the case when $\pi(a) = 7$. As $a \ge 7 + 80K\epsilon$, we have $\mathbf{x}[v] \ge 1/(2K) + 1/(4K) + 1/(8K) + 10\epsilon$. By Figure 10, $\mathbf{x}[v_1] \ge \mathbf{x}[v] - 2\epsilon$, $\mathbf{x}[v^1] = {\epsilon \over B} 1$ in the first loop and

$$\mathbf{x}[v_2] \geq \mathbf{x}[v_1] - \mathbf{x}[v_{12}] - \epsilon \geq \mathbf{x}[v] - 2\epsilon - (2^{-1}\mathbf{x}[v^1] + \epsilon) - \epsilon \\ \geq \mathbf{x}[v] - 2^{-1}(1/K + \epsilon) - 4\epsilon \geq 1/(4K) + 1/(8K) + 5\epsilon.$$

Since $\mathbf{x}[v_{21}] \leq 1/(4K) + \epsilon$ and $\mathbf{x}[v_2] - \mathbf{x}[v_{21}] > \epsilon$, we have $\mathbf{x}[v^2] =_B^{\epsilon} 1$ and

$$\mathbf{x}[v_3] \geq \mathbf{x}[v_2] - \mathbf{x}[v_{22}] - \epsilon > 1/(8K) + 2\epsilon$$

As a result, $\mathbf{x}[v_3] - \mathbf{x}[v_{31}] > \epsilon$ and $\mathbf{x}[v^3] =_B^{\epsilon} 1$.

Next, we consider the general case that $t < \pi(a) < t + 1$ for $0 \le t \le 6$. Let $b_1 b_2 b_3$ be the binary representation of t. As a is well-positioned, we have

$$b_1/(2K) + b_2/(4K) + b_3/(8K) + 10\epsilon \le \mathbf{x}[v] \le b_1/(2K) + b_2/(4K) + (b_3 + 1)/(8K) - 10\epsilon.$$

With similar arguments, after the first loop one can show that $\mathbf{x}[v^1] = {\epsilon \atop B} b_1$ and

$$b_2/(4K) + b_3/(8K) + 5\epsilon \leq \mathbf{x}[v_2] \leq b_2/(4K) + (b_3 + 1)/(8K) - 5\epsilon.$$

After the second loop, we have $\mathbf{x}[v^2] =_B^{\epsilon} b_2$ and

$$b_3/(8K) + 2\epsilon \leq \mathbf{x}[v_3] \leq (b_3 + 1)/(8K) - 2\epsilon.$$

Thus, $\mathbf{x}[v^3] =_B^{\epsilon} b_3$.

Next, we introduce COLORINGSIMULATION. Suppose $S = (V, \mathcal{T})$ is a generalized circuit with |V| = K. Let $\{v_i\}_{i \in [1:n]}$ be *n* nodes in *V*, and $\{v_i^+, v_i^-\}_{i \in [1:n]} \subset V$ be 2*n* unused nodes. We use $\mathbf{p} \in \mathbb{R}^n_+$ to denote the point encoded by nodes $\{v_i\}_{i \in [1:n]}$, that is, $p_i = 8K\mathbf{x}[v_i]$. Imagine that \mathbf{p} is a point in $S = \{\mathbf{p}^k : 1 \leq i \leq n^3\}$. COLORINGSIMULATION $(S, \{v_i\}_{i \in [1:n]}, \{v_i^+, v_i^-\}_{i \in [1:n]})$ simulates circuit *C* on input $\pi(\mathbf{p})$, by inserting the following gates into S:

- 1. Pick 3n unused nodes $\{v_{i,j}\}_{i \in [1:n], j \in [1:3]}$ in V. Call EXTRACTBITS $(\mathcal{S}, v_t, v_{t,1}, v_{t,2}, v_{t,3})$, for each $1 \le t \le n$;
- 2. View the values of $\{v_{i,j}\}$ as 3n input bits of C. Insert the corresponding logic gates from $\{G_{\vee}, G_{\wedge}, G_{\neg}\}$ into S to simulate the evaluation of C, one for each gate in C, and place the 2n output bits in $\{v_i^+, v_i^-\}$.

We obtain the following lemma for COLORINGSIMULATION $(S, \{v_i\}_{i \in [1:n]}, \{v_i^+, v_i^-\}_{i \in [1:n]})$ as a direct consequence of Lemma 8.3.

Lemma 8.4 (Point Coloring). Let S' be the generalized circuit obtained after calling the above COLORINGSIMULATION, and \mathbf{x} be an ϵ -approximate solution to S'. We let $\mathbf{p} \in \mathbb{R}^n_+$ denote the point with $p_i = 8K\mathbf{x}[v_i]$ for all $i \in [1:n]$, and $\mathbf{q} = \pi(\mathbf{p})$. We use $\{\Delta_i^+[\mathbf{q}], \Delta_i^-[\mathbf{q}]\}_{i \in [1:n]}$ to denote the 2n output bits of C evaluated at \mathbf{q} . If \mathbf{p} is a well-positioned point, then $\mathbf{x}[v_i^+] =_B^{\epsilon} \Delta_i^+[\mathbf{q}]$ and $\mathbf{x}[v_i^-] =_B^{\epsilon} \Delta_i^-[\mathbf{q}]$ for all $i \in [1:n]$.

Note that if the point **p** in the lemma above is not well-positioned, then the values of $\{v_i^+, v_i^-\}$ could be arbitrary. However, according to the definition of generalized circuits, **x** must satisfy

$$0 \le \mathbf{x}[v_i^+], \mathbf{x}[v_i^-] \le 1/K + \epsilon, \quad \forall \ i \in [1:n].$$

Finally, we build the promised generalized circuit \mathcal{S}^U with a four-step construction. At the beginning, $\mathcal{S}^U = (V, \emptyset)$ and |V| = K.

Part 1: [Equiangle Sampling Segment]

Let $\{v_i^k\}_{1 \le k \le n^3, 1 \le i \le n}$ be n^4 nodes in V. We insert G_{ζ} gates, with properly chosen parameters, and G_+ gates into \mathcal{S}^U to ensure that every ϵ -approximate solution \mathbf{x} of \mathcal{S}^U satisfies

$$\mathbf{x}[v_i^k] = \min\left(\mathbf{x}[v_i^1] + (k-1)/(8K^2), 1/K\right) \pm O(\epsilon),$$
(8)

for all $k \in [1:n^3]$ and $i \in [1:n]$.

Part 2: [Point Coloring] Pick $2n^4$ unused nodes $\{v_i^{k+}, v_i^{k-}\}_{i \in [1:n], k \in [1:n^3]}$ from V. For every $k \in [1:n^3]$, we call

COLORINGSIMULATION
$$(\mathcal{S}^U, \{v_i^k\}, \{v_i^{k+}, v_i^{k-}\}_{i \in [1:n]}).$$

Part 3: [Summing up the Coloring Vectors]

Pick 2n unused nodes $\{v_i^+, v_i^-\}_{i \in [1:n]} \subset V$. Insert properly-valued $G_{\times \zeta}$ gates and G_+ gates to ensure in the resulting generalized circuit \mathcal{S}^U , each ϵ -approximate solution \mathbf{x} satisfies

$$\mathbf{x}[v_i^+] = \sum_{1 \le k \le n^3} \left(\frac{1}{K} \mathbf{x}[v_i^{k+1}] \right) \pm O(n^3 \epsilon) \quad \text{and} \quad \mathbf{x}[v_i^-] = \sum_{1 \le k \le n^3} \left(\frac{1}{K} \mathbf{x}[v_i^{k-1}] \right) \pm O(n^3 \epsilon).$$

Part 4: [Closing the Loop]

For each $i \in [1:n]$, pick unused nodes $v'_i, v''_i \in V$ and insert the following gates:

INSERT
$$\left(\mathcal{S}^{U}, (G_{+}, v_{i}^{1}, v_{i}^{+}, v_{i}^{\prime}, nil)\right)$$
, INSERT $\left(\mathcal{S}^{U}, (G_{-}, v_{i}^{\prime}, v_{i}^{-}, v_{i}^{\prime\prime}, nil)\right)$
and INSERT $\left(\mathcal{S}^{U}, (G_{-}, v_{i}^{\prime\prime}, nil, v_{i}^{1}, nil)\right)$.

8.3 Analysis of the Reduction

We now prove the correctness of our construction.

Let \mathbf{x} be an ϵ -approximate solution to \mathcal{S}^U . Let $S = \{\mathbf{p}^k, \text{with } p_i^k = 8K\mathbf{x}[v_i^k], 1 \le k \le n^3\}$ be the set of n^3 points that we want to produce from \mathbf{x} . Let $I_G = I_G(S)$ and $I_B = I_B(S)$. For each $t \in I_G$, let $c_t \in [1 : n + 1]$ be the color of point $\mathbf{q}^t = \pi(\mathbf{p}^t)$ assigned by C, and for each $i \in [1 : n + 1]$, let $W_i = |\{t \in I_G \mid c_t = i\}|$.

It suffices to prove, as $Q = \{\pi(\mathbf{p}^k), k \in I_G\}$ can be computed in polynomial time, that Q is a panchromatic simplex of C. The line of the proof is very similar to the one for Lemma 8.2. First, we use the constraints introduced by the gates in **Part 1** to prove the following two lemmas:

Lemma 8.5 (Not Too Many Poorly-Positioned Points). $|I_B| \le n$, and hence $|I_G| \ge n^3 - n$.

Proof. For each $t \in I_B$, according to the definition of poorly-positioned points, there exists an integer $1 \leq l \leq n$ such that p_l^t is a poorly-positioned number. We will prove that, for every integer $1 \leq l \leq n$, there exists at most one $t \in [1:n^3]$ such that $p_l^t = 8K\mathbf{x}[v_l^t]$ is poorly-positioned, which implies $|I_B| \leq n$ immediately.

Assume p_l^t and $p_l^{t'}$ are both poorly-positioned, for a pair of integers $1 \le t < t' \le n^3$. Then, from the definition, there exists a pair of integers $0 \le k, k' \le 7$,

$$\left|\mathbf{x}[v_l^t] - k/(8K)\right| \le 10\epsilon$$
 and $\left|\mathbf{x}[v_l^{t'}] - k'/(8K)\right| \le 10\epsilon.$ (9)

Because (9) implies that $\mathbf{x}[v_l^t] < 1/K - \epsilon \leq \mathbf{x}_C[v_l^t]$ and $\mathbf{x}[v_l^{t'}] < 1/K - \epsilon \leq \mathbf{x}_C[v_l^{t'}]$, by Equation (8) of **Part 1**, we have

$$\mathbf{x}[v_l^t] = \mathbf{x}[v_l^1] + (t-1)/(8K^2) \pm O(\epsilon) \quad \text{and} \quad \mathbf{x}[v_l^{t'}] = \mathbf{x}[v_l^1] + (t'-1)/(8K^2) \pm O(\epsilon).$$
(10)

Hence, $\mathbf{x}[v_l^t] < \mathbf{x}[v_l^{t'}], k \leq k'$ and

$$\mathbf{x}[v_l^{t'}] - \mathbf{x}[v_l^t] = (t' - t)/(8K^2) \pm O(\epsilon)$$
(11)

Note that when k = k', Equation (9) implies that $\mathbf{x}[v_l^{t'}] - \mathbf{x}[v_l^t] \leq 20\epsilon$, while when k < k', it implies that $\mathbf{x}[v_l^{t'}] - \mathbf{x}[v_l^t] \geq (k' - k)/(8K) - 20\epsilon \geq 1/(8K) - 20\epsilon$. In both cases, we derived an inequality that contradicts (11). Thus, only one of p_l^t or p_l^t can be poorly-positioned.

Lemma 8.6 (Accommodated). $Q = \{\pi(\mathbf{p}^k), k \in I_G\}$ is accommodated and $|Q| \leq n+1$.

Proof. To show Q is accommodated, it is sufficient to prove

$$q_l^t \le q_l^{t'} \le q_l^t + 1, \qquad \text{for all } l \in [1:n] \text{ and } t, t' \in I_G \text{ such that } t < t'.$$
(12)

For the sake of contradiction, we assume that (12) is not true. We need to consider the following two cases.

First, assume $q_l^t > q_l^{t'}$ for some $t, t' \in I_G$ with t < t'. Since $q_l^{t'} < q_l^t \le 7$, we have $p_l^{t'} < 7$ and thus, $\mathbf{x}[v_l^{t'}] < 7/(8K)$. As a result, the first component of the min operator in (8) is the smallest for both t and t', implying that $\mathbf{x}[v_l^t] < \mathbf{x}[v_l^{t'}]$ and $p_l^t < p_l^{t'}$. This contradicts the assumption that $q_l^t > q_l^{t'}$.

Second, assume $q_l^{t'} - q_l^t \ge 2$ for some $t, t' \in I_G$ with t < t'. From the definition of π , we have

 $p_l^{t'} - p_l^t > 1$ and thus, $\mathbf{x}[v_l^{t'}] - \mathbf{x}[v_l^t] > 1/(8K)$. But from (8), we have

$$\mathbf{x}[v_l^{t'}] - \mathbf{x}[v_l^t] \le (t' - t)/(8K^2) + O(\epsilon) < n^3/(8K^2) + O(\epsilon) \ll 1/(8K).$$

As a result, (12) is true.

Next, we prove $|Q| \leq n + 1$. Note that the definition of Q together with (12) implies that there exist integers $t_1 < t_2 < ... < t_{|Q|} \in I_G$ such that \mathbf{q}^{t_i} is strictly dominated by $\mathbf{q}^{t_{i+1}}$, that is, $\mathbf{q}^{t_i} \neq \mathbf{q}^{t_{i+1}}$ and $q_i^{t_i} \leq q_i^{t_{i+1}}$ for all $j \in [1:n]$.

On the one hand, for every $1 \leq l \leq |Q| - 1$, there exists an integer $1 \leq k_l \leq n$ such that $q_{k_l}^{t_{l+1}} = q_{k_l}^{t_l} + 1$. On the other hand, for every $1 \leq k \leq n$, (12) implies that there is at most one $1 \leq l \leq |Q| - 1$ such that $q_k^{t_{l+1}} = q_k^{t_l} + 1$. Therefore, $|Q| \leq n + 1$.

The construction in **Part 2** and Lemma 8.4 guarantees that:

Lemma 8.7 (Correct Encoding of Colors). For each $1 \le k \le n^3$, let \mathbf{r}^k denote the vector that satisfies $r_i^k = \mathbf{x}[v_i^{k+}] - \mathbf{x}[v_i^{k-}], \forall i \in [1:n]$. For each $t \in I_G$, $\mathbf{r}^t = K\mathbf{z}^{c_t} \pm 2\epsilon$; for each $t \in I_B$, $\|\mathbf{r}^t\|_{\infty} \le 1/K + 2\epsilon$.

Recall that **Part 3** sums up these n^3 vectors $\{\mathbf{r}^k\}$. Let \mathbf{r} denote the vector that satisfies $r_i = \mathbf{x}[v_i^+] - \mathbf{x}[v_i^-]$, for all $i \in [1:n]$. Ideally, with (the constraints of) the gates inserted in **Part 4**, we wish to establish $\|\mathbf{r}\|_{\infty} = O(\epsilon)$. However, whether or not this condition holds depends on the values of $\{v_i^1\}_{1 \leq i \leq n}$ in \mathbf{x} , as the gate (G_-, a, b, c, nil) requires c = max(a - b, 0) within a difference of ϵ . For example, in the case when $\mathbf{x}[v_i^1] = 0$, the magnitude of $\mathbf{x}[v_i^-]$ could be much larger than that of $\mathbf{x}[v_i^+]$. We are able to establish the following lemma which is sufficient to carry out the correctness proof of our reduction.

Lemma 8.8 (Well-Conditioned Solution). For all $i \in [1:n]$,

- 1. if $\mathbf{x}[v_i^1] > 4\epsilon$, then $r_i = \mathbf{x}[v_i^+] \mathbf{x}[v_i^-] > -4\epsilon$; and
- 2. if $\mathbf{x}[v_i^1] < 1/K 2n^3/K^2$, then $r_i = \mathbf{x}[v_i^+] \mathbf{x}[v_i^-] < 4\epsilon$.

Proof. In order to set up a proof-by-contradiction of the first if-statement, we assume there exists some *i* such that $\mathbf{x}[v_i^1] > 4\epsilon$ and $\mathbf{x}[v_i^+] - \mathbf{x}[v_i^-] \leq -4\epsilon$.

By the first gate $(G_+, v_i^1, v_i^+, v_i', nil)$ inserted in **Part 4**, we have

$$\mathbf{x}[v_i'] = \min(\mathbf{x}[v_i^1] + \mathbf{x}[v_i^+], 1/K) \pm \epsilon \le \mathbf{x}[v_i^1] + \mathbf{x}[v_i^+] + \epsilon \le \mathbf{x}[v_i^1] + \mathbf{x}[v_i^-] - 3\epsilon.$$
(13)

By the second gate $(G_{-}, v'_{i}, v^{-}_{i}, v''_{i}, nil)$, we have

$$\mathbf{x}[v_i''] \le \max(\mathbf{x}[v_i'] - \mathbf{x}[v_i^-], 0) + \epsilon \le \max(\mathbf{x}[v_i^1] - 3\epsilon, 0) + \epsilon \le \mathbf{x}[v_i^1] - 2\epsilon,$$
(14)

where the last inequality follows from the assumption that $\mathbf{x}[v_i^1] > 4\epsilon$. Since $\mathbf{x}[v_i^1] \le 1/K + \epsilon$, we have $\mathbf{x}[v_i''] \le \mathbf{x}[v_i^1] - 2\epsilon \le 1/K - \epsilon < 1/K$. So, by the last gate $(G_{=}, v_i'', nil, v_i^1, nil)$, we have $\mathbf{x}[v_i^1] = \min(\mathbf{x}[v_i''], 1/K) \pm \epsilon = \mathbf{x}[v_i''] \pm \epsilon$, which contradicts (14). Similarly, to prove the second if-statement, we assume there exists some $1 \le i \le n$ such that $\mathbf{x}[v_i^1] < 1/K - 2n^3/K^2$ and $\mathbf{x}[v_i^+] - \mathbf{x}[v_i^-] \ge 4\epsilon$ in order to derive a contradiction.

By Part 3, $\mathbf{x}[v_i^+] \leq n^3/K^2 + O(n^3\epsilon)$. Together with the assumption, we have $\mathbf{x}[v_i^1] + \mathbf{x}[v_i^+] \leq 1/K - n^3/K^2 + O(n^3\epsilon) < 1/K$. Thus, by the first gate G_+ , we have

$$\mathbf{x}[v_i'] = \min(\mathbf{x}[v_i^1] + \mathbf{x}[v_i^+], 1/K) \pm \epsilon = \mathbf{x}[v_i^1] + \mathbf{x}[v_i^+] \pm \epsilon \ge \mathbf{x}[v_i^1] + \mathbf{x}[v_i^-] + 3\epsilon$$

and $\mathbf{x}[v_i'] \leq \mathbf{x}[v_i^1] + \mathbf{x}[v_i^+] + \epsilon \leq 1/K - n^3/K^2 + O(n^3\epsilon)$. By the second gate G_- ,

$$\mathbf{x}[v_i''] \ge \min(\mathbf{x}[v_i'] - \mathbf{x}[v_i^-], 1/K) - \epsilon = \mathbf{x}[v_i'] - \mathbf{x}[v_i^-] - \epsilon \ge \mathbf{x}[v_i^1] + 2\epsilon.$$
(15)

We also have $\mathbf{x}[v_i''] \leq \max(\mathbf{x}[v_i'] - \mathbf{x}[v_i^-], 0) + \epsilon \leq \mathbf{x}[v_i'] + \epsilon < 1/K$. However, the last gate $G_=$ implies $\mathbf{x}[v_i^1] = \min(\mathbf{x}[v_i''], 1/K) \pm \epsilon = \mathbf{x}[v_i''] \pm \epsilon$, which contradicts (15).

Now, we show that Q is a panchromatic simplex of C. By Lemma 8.6, it suffices to prove that $W_i > 0$, for all $i \in [1 : n + 1]$.

By Part 3 of the construction and Lemma 8.7,

$$\mathbf{r} = \frac{1}{K} \sum_{1 \le i \le n^3} \mathbf{r}^i \pm O(n^3 \epsilon) = \frac{1}{K} \sum_{i \in I_G} \mathbf{r}^i + \frac{1}{K} \sum_{i \in I_B} \mathbf{r}^i \pm O(n^3 \epsilon)$$
$$= \sum_{i \in I_G} \mathbf{z}^{c_i} + \frac{1}{K} \sum_{i \in I_B} \mathbf{r}^i \pm O(n^3 \epsilon) = \sum_{1 \le i \le n+1} W_i \, \mathbf{z}^i + \frac{1}{K} \sum_{i \in I_B} \mathbf{r}^i \pm O(n^3 \epsilon)$$
$$= \mathbf{r}^G + \mathbf{r}^B \pm O(n^3 \epsilon).$$

where $\mathbf{r}^G = \sum_{1 \leq i \leq n+1} W_i \mathbf{z}^i$ and $\mathbf{r}^B = \sum_{i \in I_B} \mathbf{r}^i / K$. Since $|I_B| \leq n$ and $||\mathbf{r}^i||_{\infty} \leq 1/K + \epsilon$ for each $i \in I_B$, we have $||\mathbf{r}^B||_{\infty} = O(n/K^2)$.

As $|I_G| \ge n^3 - n$, we have $\sum_{1 \le i \le n+1} W_i \ge n^3 - n$. The next lemma shows that, if one of W_i is equal to zero, then $\|\mathbf{r}^G\|_{\infty} \gg \|\mathbf{r}^B\|_{\infty}$.

Lemma 8.9. If one of W_i is equal to zero, then $\|\mathbf{r}^G\|_{\infty} \ge n^2/(3K^2)$, and thus $\|\mathbf{r}\|_{\infty} \gg 4\epsilon$.

Proof. We divide the proof into two cases. First, assume $W_{n+1} = 0$. Let $l \in [1:n]$ be the integer such that $W_l = \max_{1 \le i \le n} W_i$, then we have $W_l > n^2 - 1$. Thus, $r'_l = W_l/K \ge (n^2 - 1)/K > n^2/(3K^2)$.

Second, assume $W_t = 0$ for some $1 \le t \le n$. We have the following two cases:

- $W_{n+1} \ge n^2/2$: $r'_t = -W_{n+1}/K \le -n^2/(2K^2) < -n^2/(3K^2)$.
- $W_{n+1} < n^2/2$: Let l be the integer such that $W_l = \max_{1 \le i \le n+1} W_i$, then $l \ne t, n+1$ and $W_l > n^2 1$. Then, $r'_l = (W_l W_{n+1})/K > (n^2/2 1)/K^2 > n^2/(3K^2)$.

Therefore, if Q is not a panchromatic simplex, then one of the W_i 's is equal to zero, and hence $\|\mathbf{r}\|_{\infty} \gg 4\epsilon$. Had **Part 4** of our construction guaranteed that $\|\mathbf{r}\|_{\infty} = O(\epsilon)$, we would have completed the proof. As it is not always the case, we prove the following lemma so that we can use Lemma 8.8 to complete the proof.

Lemma 8.10 (Well-Conditioned). For all $i \in [1:n]$, $4\epsilon < \mathbf{x}[v_i^1] < 1/K - 2n^3/K^2$.

Proof. In this proof, we will use the following boundary condition of circuit C: For each $\mathbf{q} \in B^n$ and $1 \le k \ne l \le n$,

- **B.1:** if $q_k = 0$, then $\operatorname{Color}_C[\mathbf{q}] \neq n+1$;
- **B.2:** if $q_k = 0$ and $q_l > 0$, then $\operatorname{Color}_C[\mathbf{q}] \neq l$;
- **B.3:** if $q_k = 7$, then $\operatorname{Color}_C[\mathbf{q}] \neq k$; and
- **B.4:** if $q_k = 7$ and $\operatorname{Color}_C[\mathbf{q}] = l \neq k$, then $q_l = 0$.

These conditions follow directly from the definition of valid circuits. Recall $1/K = 2^{-6m}$, $\epsilon = 2^{-18m} = 1/K^3$ and $2^m > n$.

First, if there exists an integer $k \in [1:n]$ such that $\mathbf{x}[v_k^1] \leq 4\epsilon$, then $q_k^t = 0$ for all $t \in I_G$. By B.1, $W_{n+1} = 0$. Let l be the integer such that $W_l = \max_{1 \leq i \leq n} W_i$. As $\sum_{i=1}^{n+1} W_i = |I_G| \geq n^3 - n$, we have $W_l \geq n^2 - 1$. So, $r_l \geq W_l/K^2 - O(n/K^2) - O(n^3\epsilon) \gg 4\epsilon$. Now consider the following two cases:

- If $\mathbf{x}[v_l^1] < 1/K 2n^3/K^2$, then we get a contradiction in Lemma 8.8.
- If $\mathbf{x}[v_I^1] \ge 1/K 2n^3/K^2$, then for all $t \in I_G$,

$$p_l^t = 8K \Big(\min \left(\mathbf{x}[v_l^1] + (t-1)/(8K^2), 1/K \right) \pm O(\epsilon) \Big) > 1$$

and hence $q_l^t > 0$. By B.2, we have $W_l = 0$, contradicting the assumption.

Second, if there exists an integer $k \in [1 : n]$ such that $\mathbf{x}[v_k^1] \ge 1/K - 2n^3/K^2$, then for all $t \in I_G$, we have $q_k^t = 7$. By B.3, $W_k = 0$. If $W_{n+1} \ge n^2/2$, then

$$r_k \le -W_{n+1}/K^2 + O(n/K^2) + O(n^3\epsilon) \ll -4\epsilon,$$

which contradicts the assumption that $\mathbf{x}[v_k^1] \ge 1/K - 2n^3/K^2 > 4\epsilon$ (see Lemma 8.8.1). Below, we assume $W_{n+1} < n^2/2$.

Let *l* be the integer such that $W_l = \max_{1 \le i \le n+1} W_i$. Since $W_k = 0$, we have $W_l \ge n^2 - 1$ and $l \ne k$. As $W_{n+1} < n^2/2$, $W_l - W_{n+1} > n^2/2 - 1$ and thus,

$$r_l \ge (W_l - W_{n+1})/K^2 - O(n/K^2) - O(n^3\epsilon) \gg 4\epsilon.$$

We now consider the following two cases:

- If $\mathbf{x}[v_l^1] < 1/K 2n^3/K^2$, then we get a contradiction in Lemma 8.8.2;
- If $\mathbf{x}[v_l^1] \ge 1/K 2n^3/K^2$, then $p_l^t > 1$ and thus $q_l^t > 0$ for all $t \in I_G$. By B.4, we have $W_l = 0$ which contradicts the assumption.

9 Extensions and Open Problems

9.1 Sparse Games are Hard

As fixed points and Nash equilibria are fundamental to many other search and optimization problems, our results and techniques may have a broader scope of applications and implications. So far, our complexity results on the computation and approximation of Nash equilibria have been extended to Arrow-Debreu equilibria [28]. They can also be naturally extended to both r-player games [45] and r-graphical games [33], for every fixed $r \geq 3$. Since the announcement of our work, it has been shown that the Nash equilibrium is **PPAD**-hard to approximate in fully polynomial time even for bimatrix games with some special payoff structures, such as bimatrix games in which all payoff entries are either 0 or 1 [14], or in which most of the payoff entries are 0. In the latter case, we can strengthen our gadgets to prove the following theorem:

Theorem 9.1 (SPARSE BIMATRIX). Nash equilibria remains **PPAD**-hard to approximate in fully polynomial time for sparse bimatrix games in which each row and column of the two payoff matrices contains at most 10 nonzero entries.

The reduction needed in proving this theorem is similar to the one used in proving Theorem 5.1. The key difference is that we first reduce BROUWER^{f_1} to a sparse generalized circuit, where a generalized circuit is *sparse* if each node is used by at most two gates as their input nodes. We then refine our gadget games for G_{ζ} , G_{\wedge} or G_{\vee} , to guarantee that the resulting bimatrix game is sparse. Details of the proof can be found in [11].

9.2 Open Questions and Conjectures

There remains a complexity gap in the approximation of two-player Nash equilibria: Lipton, Markakis and Mehta [40] show that an ϵ -approximate Nash equilibrium can be computed in $n^{O(\log n/\epsilon^2)}$ -time, while this paper shows that, for ϵ of order 1/poly(n), no algorithm can find an ϵ -approximate Nash equilibrium in $\text{poly}(n, 1/\epsilon)$ -time, unless **PPAD** is contained in **P**. However, our hardness result does not cover the case when ϵ is a constant between 0 and 1, or of order 1/polylog(n). Naturally, it is unlikely that finding an ϵ -approximate Nash equilibrium is **PPAD**complete when ϵ is an absolute constant, for otherwise, all search problems in **PPAD** would be solvable in $n^{O(\log n)}$ -time, due to the result of [40].

Thinking optimistically, we would like to see the following conjectures turn out to be true.

Conjecture 1 (PTAS for BIMATRIX). There is an $O(n^{k+\epsilon^{-c}})$ -time algorithm for finding an ϵ -approximate Nash equilibrium in a two-player game, for some constants c and k.

Conjecture 2 (Smoothed BIMATRIX). There is an algorithm for BIMATRIX with smoothed complexity $O(n^{k+\sigma^{-c}})$ under perturbations with magnitude σ , for some constants c and k.

For a sufficiently large ϵ , such as $\epsilon \ge 1/2$, an ϵ -approximate Nash equilibrium can be found in polynomial time [19, 36], by examining small-support strategies. However, new techniques are needed to prove Conjecture 1 [22]. Lemma 2.2 implies that Conjecture 1 is true for ϵ -wellsupported Nash equilibrium if and only if it is true for ϵ -approximate Nash equilibrium. For each bimatrix game (**A**, **B**) such that rank(**A** + **B**) is a constant, Kannan and Theobald [31] found a fully-polynomial-time algorithm to approximate Nash equilibria.

For Conjecture 2, one might be able to prove a weaker version of this conjecture by extending the analysis of [4] to show that there is an algorithm for BIMATRIX with smoothed complexity $n^{O(\log n/\sigma^2)}$. We also conjecture that Corollary 5.4 remains true without any complexity assumption on **PPAD**. A positive answer would extend the result of Savani and von Stengel [50] to smoothed bimatrix games. Another interesting question is whether the average-case complexity of the Lemke-Howson algorithm is polynomial.

Of course, the fact that two-player Nash equilibria and Arrow-Debreu equilibria are **PPAD**hard to compute in the smoothed model does not necessarily imply that game and market problems are hard to solve in practice. In addition to possible noise and imprecision in inputs, practical problems might have other special structure that makes equilibrium computation or approximation more tractable. The game and market problems and their hardness results might provide an opportunity and a family of concrete problems for discovering new input models that can help us to rigorously evaluate the performance of practical equilibrium algorithms and heuristics.

Theorem 5.2 implies that for any r > 2, the computation of an *r*-player Nash equilibrium can be reduced in polynomial time to the computation of a two-player Nash equilibrium. However, the implied reduction is not very natural: The *r*-player Nash equilibrium problem is first reduced to END-OF-LINE, then to BROUWER, and then to BIMATRIX. It remains an interesting question to find a more direct reduction from *r*-player Nash equilibria to two-player Nash equilibria.

The following complexity question about Nash equilibria is due to Vijay Vazirani: Are the counting versions of all **PPAD** complete-problems as hard as the counting version of BIMATRIX? Gilboa and Zemel [24] showed that deciding whether a bimatrix game has a unique Nash equilibrium is **NP**-hard. Their technique was extended in [17] to prove that counting the number of Nash equilibria is #P-hard. Because the reduction between search problems only requires a many-to-one map between solutions, the number of solutions is not necessarily preserved. More restricted reductions are needed to solve Vazirani's question.

Finally, even though our results in this paper as well as the results of [18, 10, 20] provide strong evidence that equilibrium computation might be hard for \mathbf{P} , the hardness of the **PPAD** complexity class itself is largely unknown [29]. On one hand, Megiddo [41] proved that if BIMA-TRIX is **NP**-hard, then **NP** = **coNP**. On the other hand, there are oracles that separate **PPAD** from \mathbf{P} , and various discrete fixed point problems such as the computational version of Sperner's Lemma, requires an exponential number of functional evaluations in the query model, deterministic [26, 8] or randomized [13], and in the quantum query model [23, 13]. It is desirable to find stronger evidences that **PPAD** is not contained in \mathbf{P} . Does the existence of one-way functions imply that **PPAD** is not contained in \mathbf{P} ? Does "FACTORING is not in \mathbf{P} " imply that **PPAD** is not contained in \mathbf{P} ? Characterizing the hardness of the **PPAD** class is a great and a challenging problem.

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References

- [1] John Reif, Nicole Immorlica, Steve Vavasis, Christos Papadimitriou, Mohammad Mahdian, Ding-Zhu Du, Santosh Vempala, Aram Harrow, Adam Kalai, Imre Bárány, Adrian Vetta, Jonathan Kelner and a number of other people asked whether the smoothed complexity of the Lemke-Howson algorithm or Nash Equilibria is polynomial, 2001–2005.
- [2] T. Abbott, D. Kane, and P. Valiant. On the complexity of two-player win-lose games. In FOCS '05: Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, pages 113–122, 2005.
- [3] K.J. Arrow and G. Debreu. Existence of an equilibrium for a competitive economy. *Econo-metrica*, 22:265–290, 1954.
- [4] I. Bárány, S. Vempala, and A. Vetta. Nash equilibria in random games. In FOCS '05: Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, pages 123–131, 2005.
- [5] L. Blum, M. Shub, and S. Smale. On a theory of computation over the real numbers; NP completeness, recursive functions and universal machines. *Bulletin of the AMS*, 21(1):1–46, July 1989.
- [6] K.-H. Borgwardt. The average number of steps required by the simplex method is polynomial. Zeitschrift f
 ür Operations Research, 26:157–177, 1982.
- [7] L.E.J. Brouwer. Über Abbildung von Mannigfaltigkeiten. Mathematische Annalen, 71:97– 115, 1910.

- [8] X. Chen and X. Deng. On algorithms for discrete and approximate Brouwer fixed points. In STOC '05: Proceedings of the 37th Annual ACM Symposium on Theory of computing, pages 323–330, 2005.
- [9] X. Chen and X. Deng. On the complexity of 2D discrete fixed point problem. In ICALP '06: Proceedings of the 33rd International Colloquium on Automata, Languages and Programming, pages 489–500, 2006.
- [10] X. Chen and X. Deng. 3-Nash is PPAD-complete. In *Electronic Colloquium in Computational Complexity*, TR05-134, 2005.
- [11] X. Chen, X. Deng, and S.-H. Teng. Sparse games are hard. In Proceedings of the 2nd Workshop on Internet and Network Economics, pages 262–273, 2006.
- [12] X. Chen, L.-S. Huang, and S.-H. Teng. Market equilibria with hybrid linear-Leontief utilities. In Proceedings of the 2nd Workshop on Internet and Network Economics, pages 274–285, 2006.
- [13] X. Chen and S.-H. Teng. Paths beyond local search: A nearly tight bound for randomized fixed-point computation. arXiv, 2007. http://arxiv.org/abs/cs.GT/0702088.
- [14] X. Chen, S.-H. Teng, and P.A. Valiant. The approximation complexity of win-lose games. In SODA '07: Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms, 2007.
- [15] B. Codenotti, A. Saberi, K. Varadarajan, and Y. Ye. Leontief economies encode nonzero sum two-player games. In SODA '06: Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms, pages 659–667, 2006.
- [16] A. Condon, H. Edelsbrunner, E. Emerson, L. Fortnow, S. Haber, R. Karp, D. Leivant, R. Lipton, N. Lynch, I. Parberry, C. Papadimitriou, M. Rabin, A. Rosenberg, J. Royer, J. Savage, A. Selman, C. Smith, E. Tardos, and J. Vitter. Challenges for theory of computing: Report of an NSF-sponsored workshop on research in theoretical computer science. *SIGACT News*, 30(2):62–76, 1999.
- [17] V. Conitzer and T. Sandholm. Complexity results about nash equilibria. In In Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI), 2003.
- [18] C. Daskalakis, P.W. Goldberg, and C.H. Papadimitriou. The complexity of computing a Nash equilibrium. In STOC '06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, pages 71–78, 2006.
- [19] C. Daskalakis, A. Mehta, and C.H. Papadimitriou. A note on approximate Nash equilibria. In Proceedings of the 2nd Workshop on Internet and Network Economics, pages 297–306, 2006.

- [20] C. Daskalakis and C.H. Papadimitriou. Three-player games are hard. In *Electronic Collo-quium in Computational Complexity*, TR05-139, 2005.
- [21] X. Deng, C. Papadimitriou, and S. Safra. On the complexity of price equilibria. Journal of Computer and System Sciences, 67(2):311–324, 2003.
- [22] T. Feder, H. Nazerzadeh, and A. Saberi. Approximating nash equilibria using small-support strategies. Stanford, 2006.
- [23] K. Friedl, G. Ivanyos, M. Santha, and F. Verhoeven. On the black-box complexity of Sperner's lemma. In Proceedings of the 15th International Symposium on Fundamentals of Computation Theory, pages 245–257, 2005.
- [24] I. Gilboa and E. Zemel. Nash and correlated equilibria: Some complexity considerations. Games and Economic Behavior, 1(1).
- [25] P.W. Goldberg and C.H. Papadimitriou. Reducibility among equilibrium problems. In STOC '06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, pages 61–70, 2006.
- [26] M.D. Hirsch, C.H. Papadimitriou, and S. Vavasis. Exponential lower bounds for finding Brouwer fixed points. *Journal of Complexity*, 5:379–416, 1989.
- [27] C. A. Holt and A. E. Roth. The Nash equilibrium: A perspective. PNAS, 101(12):3999–4002, March 2004.
- [28] L.-S. Huang and S.-H. Teng. On the approximation and smoothed complexity of Leontief market equilibria. In *Electronic Colloquium in Computational Complexity*, TR06-031, 2006.
- [29] D. Johnson. The NP-completeness column: Finding needles in haystacks. ACM Transactions on Algorithms, (to appear), April 2007.
- [30] S. Kakutani. A generalization of Brouwer's fixed point theorem. Duke Mathematical Journal, 8:457–459, 1941.
- [31] R. Kannan and T. Theobald. Games of fixed rank: A hierarchy of bimatrix games. In SODA '07: Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms, 2007.
- [32] N. Karmarkar. A new polynomial time algorithm for linear programming. Combinatorica, 4:373–395, 1984.
- [33] M. Kearns, M. Littman, and S. Singh. Graphical models for game theory. In Proceedings of the Conference on Uncertainty in Artificial Intelligence, pages 253–260, 2001.
- [34] L.G. Khachian. A polynomial algorithm in linear programming. Doklady Akademia Nauk, SSSR 244:1093–1096, English translation in Soviet Math. Dokl. 20, 191–194, 1979.

- [35] V. Klee and G.J. Minty. How good is the simplex algorithm? In O. Shisha, editor, *Inequalities* – *III*, pages 159–175. Academic Press, 1972.
- [36] S. Kontogiannis, P. Panagopoulou, and P. Spirakis. Polynomial algorithms for approximating Nash equilibria of bimatrix games. In *Proceedings of the 2nd Workshop on Internet and Network Economics*, pages 286–296, 2006.
- [37] C.E. Lemke. Bimatrix equilibrium points and mathematical programming. Management Science, 11:681–689, 1965.
- [38] C.E. Lemke and J.T. Howson, Jr. Equilibrium points of bimatrix games. Journal of the Society for Industrial and Applied Mathematics, 12:413–423, 1964.
- [39] R.J. Leonard. Reading Cournot, reading Nash: The creation and stabilisation of the Nash equilibrium. *Economic Journal*, 104(424):492–511, 1994.
- [40] R.J. Lipton, E. Markakis, and A. Mehta. Playing large games using simple strategies. In Proceedings of the 4th ACM conference on Electronic commerce, pages 36–41, 2004.
- [41] N. Megiddo. A note on the complexity of P-matrix LCP and computing an equilibrium. Research Report RJ6439, IBM Almaden Research Center, San Jose, 1988.
- [42] N. Megiddo and C.H. Papadimitriou. On total functions, existence theorems and computational complexity. *Theoretical Computer Science*, 81:317–324, 1991.
- [43] O. Morgenstern and J. von Neumann. Theory of Games and Economic Behavior. Princeton University Press, 1947.
- [44] J. Nash. Equilibrium point in n-person games. Porceedings of the National Academy of the USA, 36(1):48–49, 1950.
- [45] J. Nash. Noncooperative games. Annals of Mathematics, 54:289–295, 1951.
- [46] C.H. Papadimitriou. On inefficient proofs of existence and complexity classes. In Proceedings of the 4th Czechoslovakian Symposium on Combinatorics, 1991.
- [47] C.H. Papadimitriou. On the complexity of the parity argument and other inefficient proofs of existence. *Journal of Computer and System Sciences*, pages 498–532, 1994.
- [48] C.H. Papadimitriou. Algorithms, games, and the internet. In STOC '01: Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, pages 749–753, 2001.
- [49] T. Sandholm. Issues in computational vickrey auctions. International Journal of Electronic Commerce, 4(3):107 – 129, March 2000.
- [50] R. Savani and B. von Stengel. Exponentially many steps for finding a Nash equilibrium in a bimatrix game. In FOCS '04: Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, pages 258–267, 2004.

- [51] H. Scarf. The approximation of fixed points of a continuous mapping. SIAM Journal on Applied Mathematics, 15:997–1007, 1967.
- [52] H. Scarf. On the computation of equilibrium prices. In W. Fellner, editor, Ten Economic Studies in the Tradition of Irving Fisher. New York: John Wiley & Sons, 1967.
- [53] E. Sperner. Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes. Abhandlungen aus dem Mathematischen Seminar Universität Hamburg, 6:265–272, 1928.
- [54] D.A. Spielman and S.-H. Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. Journal of the ACM, 51(3):385–463, 2004, also in STOC '01: Proceedings of the 33rd Annual ACM Symposium on the Theory of Computing.
- [55] D.A. Spielman and S.-H. Teng. Smoothed analysis of algorithms and heuristics: Progress and open questions. In L. Pardo, A. Pinkus, E. Süli and M.J. Todd, editor, *Foundations of Computational Mathematics*, pages 274–342. Cambridge University Press, 2006.
- [56] J. von Neumann. Zur theorie der gesellschaftsspiele. Mathematische Annalen, 100:295–320, 1928.
- [57] R. Wilson. Computing equilibria of n-person games. SIAM Journal on Applied Mathematics, 21:80–87, 1971.
- [58] Y. Ye. Exchange market equilibria with Leontief's utility: Freedom of pricing leads to rationality. In Proceedings of the 1st Workshop on Internet and Network Economics, pages 14–23, 2005.

A Perturbation and Probabilistic Approximation

In this section, we prove Lemma 3.2. To help explain the probabilistic reduction from the approximation of bimatrix games to the solution of perturbed bimatrix games, we first define the notion of many-way polynomial reductions among **TFNP** problems.

Definition A.1 (Many-way Reduction). Let \mathcal{F} be a set of polynomial-time computable functions and g be a polynomial-time computable function. A search problem SEARCH^{R_1} \in **TFNP** is (\mathcal{F} , g)reducible to SEARCH^{R_2} \in **TFNP** if, for all $y \in \{0,1\}^*$, $(f(x), y) \in R_2$ implies $(x, g(y)) \in R_1$ for every input x of R_1 and for every function $f \in \mathcal{F}$.

Proof. (of Lemma 3.2) We will only give a proof of the lemma under uniform perturbations. With a slightly more complex argument to handle the low probability case when the absolute value of the perturbation is too large, we can similarly prove the lemma under Gaussian perturbations.

Suppose J is an algorithm with polynomial smoothed complexity for BIMATRIX. Let $T_J(\mathbf{A}, \mathbf{B})$ be the complexity of J for solving the bimatrix game defined by (\mathbf{A}, \mathbf{B}) . Let $N_{\sigma}()$ denotes the uniform perturbation with magnitude σ . Then there exists constants c, k_1 and k_2 such that for all $0 < \sigma < 1$,

$$\max_{\bar{\mathbf{A}},\bar{\mathbf{B}}\in\mathbb{R}_{[-1,1]}^{n\times n}} \mathbb{E}_{\mathbf{A}\leftarrow N_{\sigma}(\bar{\mathbf{A}}),\mathbf{B}\leftarrow N_{\sigma}(\bar{\mathbf{B}})} \left[T_{J}(\mathbf{A},\mathbf{B})\right] \leq c \cdot n^{k_{1}}\sigma^{-k_{2}}$$

For each pair of perturbation matrices $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{n \times n}_{[-\sigma,\sigma]}$, we can define a function $f_{(\mathbf{S},\mathbf{T})}$: $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ as $f_{(\mathbf{S},\mathbf{T})}((\bar{\mathbf{A}},\bar{\mathbf{B}})) = (\bar{\mathbf{A}} + \mathbf{S}, \bar{\mathbf{B}} + \mathbf{T})$. Let \mathcal{F}_{σ} be the set of all such functions, i.e.,

$$\mathcal{F}_{\sigma} = \left\{ f_{(\mathbf{S},\mathbf{T})} | \mathbf{S}, \mathbf{T} \in \mathbb{R}^{n \times n}_{[-\sigma,\sigma]} \right\}.$$

Let g be the identity function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}^n$.

We now show that the problem of computing an ϵ -approximate Nash equilibrium is $(\mathcal{F}_{\epsilon/2}, g)$ reducible to the problem of finding a Nash equilibrium of perturbed instances. More specifically,
we prove that for every bimatrix game $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ and for every $f_{(\mathbf{S},\mathbf{T})} \in \mathcal{F}_{\epsilon/2}$, an Nash equilibrium (\mathbf{x}, \mathbf{y}) of $f_{(\mathbf{S},\mathbf{T})}((\bar{\mathbf{A}}, \bar{\mathbf{B}}))$ is an ϵ -approximate Nash equilibrium of $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$.

Let $\mathbf{A} = \bar{\mathbf{A}} + \mathbf{S}$ and $\mathbf{B} = \bar{\mathbf{B}} + \mathbf{T}$. Then,

$$|\mathbf{x}^T \mathbf{A} \mathbf{y} - \mathbf{x}^T \bar{\mathbf{A}} \mathbf{y}| = |\mathbf{x}^T \mathbf{S} \mathbf{y}| \le \epsilon/2$$
 (16)

$$|\mathbf{x}^T \mathbf{B} \mathbf{y} - \mathbf{x}^T \bar{\mathbf{B}} \mathbf{y}| = |\mathbf{x}^T \mathbf{T} \mathbf{y}| \le \epsilon/2.$$
(17)

Thus, for each Nash equilibrium (\mathbf{x}, \mathbf{y}) of (\mathbf{A}, \mathbf{B}) , for any $(\mathbf{x}', \mathbf{y}')$,

$$(\mathbf{x}')^T \bar{\mathbf{A}} \mathbf{y} - \mathbf{x}^T \bar{\mathbf{A}} \mathbf{y} \le ((\mathbf{x}')^T \mathbf{A} \mathbf{y} - \mathbf{x}^T \mathbf{A} \mathbf{y}) + \epsilon \le \epsilon.$$

Similarly, $\mathbf{x}^T \bar{\mathbf{B}} \mathbf{y}' - \mathbf{x}^T \bar{\mathbf{B}} \mathbf{y} \leq \epsilon$. Therefore, (\mathbf{x}, \mathbf{y}) is an ϵ -Nash equilibrium of game $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$.

Now given the algorithm J with polynomial smoothed time-complexity for BIMATRIX, we can apply the following randomized algorithm (with the help of a $(\mathcal{F}_{\epsilon/2}, g)$ -many-way reduction) to find an ϵ -approximate Nash equilibrium of game $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$:

Algorithm NashApproximationByPerturbations(A, B)

- 1. Randomly choose a pair of perturbation matrices \mathbf{S}, \mathbf{T} of magnitude σ and set $\mathbf{A} = \bar{\mathbf{A}} + \mathbf{S}$ and $\mathbf{B} = \bar{\mathbf{B}} + \mathbf{T}$.
- 2. Apply algorithm J to find a Nash equilibrium (\mathbf{x}, \mathbf{y}) of (\mathbf{A}, \mathbf{B}) .
- 3. Return (\mathbf{x}, \mathbf{y}) .

The expected time complexity of NashApproximationByPerturbation is bounded from above by the smoothed complexity of J when the magnitude perturbations is $\epsilon/2$ and hence is at most $2^{k_2}c \cdot n^{k_1}\epsilon^{-k_2}$.

B Padding Generalized Circuits: Proof of Theorem 4.7

Suppose $\mathcal{S} = (V, \mathcal{T})$ is a generalized circuit. Let K = |V|.

First, S has a $1/K^3$ -approximate solution because 1) POLY³-GCIRCUIT is reducible to POLY¹²-BIMATRIX (Section 6); and 2) every two-player game has a Nash equilibrium. Thus, the theorem is true for $c \leq 3$.

To prove the theorem for the case when c > 3, we reduce POLY^c-GCIRCUIT to POLY³-GCIRCUIT. Suppose c = 2b + 1, where b > 1. We construct a new circuit S' = (V', T') by inserting some dummy nodes into S as following:

- $V \subset V'$, $|V| = K^b > K$ and $|\mathcal{T}'| = |\mathcal{T}|$;
- For each gate $T = (G, v_1, v_2, v, \alpha) \in \mathcal{T}$, if $G \notin \{G_{\zeta}, G_{\times \zeta}\}$ (and thus, $\alpha = nil$), then $T \in \mathcal{T}'$; otherwise, gate $(G, v_1, v_2, v, K^{1-b}\alpha) \in \mathcal{T}'$.

Let \mathbf{x}' be a $1/|V'|^3$ -approximate solution of \mathcal{S}' . Note that $|V'|^3 = 1/K^{3b}$. We construct an assignment $\mathbf{x} : V \to \mathbb{R}$ by setting $\mathbf{x}[v] = K^{b-1}\mathbf{x}'[v]$ for every $v \in V$. One can easily check that \mathbf{x} is a $1/K^{2b+1}$ -approximate solution to the original circuit \mathcal{S} . We then apply $1/K^{2b+1} = 1/K^c$.

C Padding Bimatrix Games: Proof of Lemma 5.8

Let c be the constant such that POLY^c-BIMATRIX is known to be **PPAD**-complete. If c < 2, then finding an n^{-2} -approximate Nash equilibrium is harder, and thus is also complete in **PPAD**. With this, without loss of generality, we assume that $c \ge 2$. To prove the lemma, we only need to show that for every constant c' such that 0 < c' < c, POLY^c-BIMATRIX is polynomial-time reducible to POLY^{c'}-BIMATRIX.

Suppose $\mathcal{G} = (\mathbf{A}, \mathbf{B})$ is an $n \times n$ positively normalized two-player game. We transform it into a new $n \times n$ game $\mathcal{G}' = (\mathbf{A}', \mathbf{B}')$ as follows:

$$a'_{i,j} = a_{i,j} + \left(1 - \max_{1 \le k \le n} a_{k,j}\right)$$
 and $b'_{i,j} = b_{i,j} + \left(1 - \max_{1 \le k \le n} a_{i,k}\right)$, $\forall i, j : 1 \le i, j \le n$.

One can verify that any ϵ -approximate Nash equilibrium of \mathcal{G}' is also an ϵ -approximate Nash equilibrium of \mathcal{G} . Besides, every column of \mathbf{A}' and every row of \mathbf{B}' has at least one entry with value 1.

Next, we construct an $n'' \times n''$ game $\mathcal{G}'' = (\mathbf{A}'', \mathbf{B}'')$ where $n'' = n^{\frac{2c}{c'}} > n$ as follows: \mathbf{A}'' and \mathbf{B}'' are both 2×2 block matrices with $\mathbf{A}''_{1,1} = \mathbf{A}'$, $\mathbf{B}''_{1,1} = \mathbf{B}'$, $\mathbf{A}''_{1,2} = \mathbf{B}''_{2,1} = 1$ and $\mathbf{A}''_{2,1} = \mathbf{A}''_{2,2} = \mathbf{B}''_{1,2} = \mathbf{B}''_{2,2} = 0$. Now let $(\mathbf{x}'', \mathbf{y}'')$ be any $1/n''^{c'} = 1/n^{2c}$ -approximate Nash equilibrium of game $\mathcal{G}'' = (\mathbf{A}'', \mathbf{B}'')$. By the definition of ϵ -approximate Nash equilibria, one can show that $0 \leq \sum_{n < i \leq n''} x''_i, \sum_{n < i \leq n''} y''_i \leq n^{1-2c} \ll 1/2$, since we assumed that $c \geq 2$. Let $a = \sum_{1 \leq i \leq n} x''_i$ and $b = \sum_{1 \leq i \leq n} y''_i$. We construct a profile of mixed strategies $(\mathbf{x}', \mathbf{y}')$ of \mathcal{G}' as follows: $x'_i = x''_i/a$ and $y'_i = y''_i/b$ for all $i \in [1:n]$. Since a, b > 1/2, one can show that $(\mathbf{x}', \mathbf{y}')$ is a $2/n^{2c}$ -approximate Nash equilibrium of \mathcal{G}' , which is also a $1/n^c$ -approximate Nash equilibrium of the original game \mathcal{G} .

D Gadget Gates: Complete the Proof of Lemma 6.4

Proof for G_{ζ} *Gates.* From (1), (2) and Figure 3, we have

$$\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \rangle - \langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \rangle = \overline{\mathbf{x}}[v] - \alpha, \text{ and}$$

 $\langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \rangle - \langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \rangle = (\overline{\mathbf{y}}_{C}[v] - \overline{\mathbf{y}}[v]) - \overline{\mathbf{y}}[v].$

If $\overline{\mathbf{x}}[v] > \alpha + \epsilon$, then from the first equation, we have $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$. But the second equation implies $\overline{\mathbf{x}}[v] = 0$, which contradicts our assumption that $\overline{\mathbf{x}}[v] > 0$.

If $\overline{\mathbf{x}}[v] < \alpha - \epsilon$, then from the first equation, we have $\overline{\mathbf{y}}[v] = 0$. But the second equation implies that $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v] \ge 1/K - \epsilon$, which contradicts the assumption that $\overline{\mathbf{x}}[v] < \alpha - \epsilon$ and $\alpha \le 1/K$.

Proof for $G_{\times\zeta}$ *Gates.* From (1), (2) and Figure 3, we have

$$\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \rangle - \langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \rangle = \alpha \, \overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v], \text{ and} \langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \rangle - \langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \rangle = \overline{\mathbf{y}}[v] - \left(\overline{\mathbf{y}}_C[v] - \overline{\mathbf{y}}[v] \right).$$

If $\overline{\mathbf{x}}[v] > \min(\alpha \overline{\mathbf{x}}[v_1], 1/K) + \epsilon$, then $\overline{\mathbf{x}}[v] > \alpha \overline{\mathbf{x}}[v_1] + \epsilon$, since $\overline{\mathbf{x}}[v] \le \overline{\mathbf{x}}_C[v] \le 1/K + \epsilon$. By the first equation, we have $\overline{\mathbf{y}}[v] = 0$ and the second one implies that $\overline{\mathbf{x}}[v] = 0$, which contradicts the assumption that $\overline{\mathbf{x}}[v] > \min(\alpha \overline{\mathbf{x}}[v_1], 1/K) + \epsilon > 0$.

If $\overline{\mathbf{x}}[v] < \min(\alpha \overline{\mathbf{x}}[v_1], 1/K) - \epsilon \le \alpha \overline{\mathbf{x}}[v_1] - \epsilon$, then the first equation shows $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$ and thus by the second equation, we have $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v] \ge 1/K - \epsilon$, which contradicts the assumption that $\overline{\mathbf{x}}[v] < \min(\alpha \overline{\mathbf{x}}[v_1], 1/K) - \epsilon \le 1/K - \epsilon$.

Proof for $G_{=}$ Gates. $G_{=}$ is a special case of $G_{\times\zeta}$, with parameter $\alpha = 1$.

Proof for G_{-} *Gates.* From (1), (2) and Figure 3, we have

$$\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \rangle - \langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \rangle = \overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2] - \overline{\mathbf{x}}[v], \text{ and} \langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \rangle - \langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \rangle = \overline{\mathbf{y}}[v] - \left(\overline{\mathbf{y}}_C[v] - \overline{\mathbf{y}}[v] \right).$$

If $\overline{\mathbf{x}}[v] > \max(\overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2], 0) + \epsilon \ge \overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2] + \epsilon$, then the first equation implies $\overline{\mathbf{y}}[v] = 0$. By the second equation, we have $\overline{\mathbf{x}}[v] = 0$ which contradicts the assumption that $\overline{\mathbf{x}}[v] > \max(\overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2], 0) + \epsilon > 0$.

If $\overline{\mathbf{x}}[v] < \min(\overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2], 1/K) - \epsilon \leq \overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2] - \epsilon$, then by the first equation, we have $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$. By the second equation, we have $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v] \geq 1/K - \epsilon$, contradicting the assumption that $\overline{\mathbf{x}}[v] < \min(\overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2], 1/K) - \epsilon \leq 1/K - \epsilon$.

Proof for G_{\leq} *Gates.* From (1), (2) and Figure 3, we have

$$\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \rangle - \langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \rangle = \overline{\mathbf{x}}[v_1] - \overline{\mathbf{x}}[v_2], \text{ and} \langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \rangle - \langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \rangle = \left(\overline{\mathbf{y}}_C[v] - \overline{\mathbf{y}}[v] \right) - \overline{\mathbf{y}}[v].$$

If $\overline{\mathbf{x}}[v_1] < \overline{\mathbf{x}}[v_2] - \epsilon$, then $\overline{\mathbf{y}}[v] = 0$ according to the first equation. By the second equation, we have $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v] = 1/K \pm \epsilon$ and thus, $\overline{\mathbf{x}}[v] = \frac{\epsilon}{B} 1$.

If $\overline{\mathbf{x}}[v_1] > \overline{\mathbf{x}}[v_2] + \epsilon$, then $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$ according to the first equation. By the second one, we have $\overline{\mathbf{x}}[v] = 0$ and thus, $\overline{\mathbf{x}}[v] = \overset{\epsilon}{B} 0$.

Proof for G_{\vee} *Gates.* From (1), (2) and Figure 3, we have

$$\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \rangle - \langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \rangle = \overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] - 1/(2K), \text{ and} \langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \rangle - \langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \rangle = \overline{\mathbf{y}}[v] - (\overline{\mathbf{y}}_C[v] - \overline{\mathbf{y}}[v]).$$

If $\overline{\mathbf{x}}[v_1] =_B^{\epsilon} 1$ or $\overline{\mathbf{x}}[v_2] =_B^{\epsilon} 1$, then $\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] \ge 1/K - \epsilon$. By the first equation $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$. By the second equation, we have $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v] = 1/K \pm \epsilon$ and thus, $\overline{\mathbf{x}}[v] =_B^{\epsilon} 1$.

If $\overline{\mathbf{x}}[v_1] =_B^{\epsilon} 0$ and $\overline{\mathbf{x}}[v_2] =_B^{\epsilon} 0$, then $\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] \le 2\epsilon$. From the first equation, $\overline{\mathbf{y}}[v] = 0$. Then, the second equation implies $\overline{\mathbf{x}}[v] =_B^{\epsilon} 0$.

Proof for G_{\wedge} *Gates.* From (1), (2) and Figure 3, we have

$$\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \rangle - \langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \rangle = \overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] - 3/(2K), \text{ and} \langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \rangle - \langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \rangle = \overline{\mathbf{y}}[v] - (\overline{\mathbf{y}}_C[v] - \overline{\mathbf{y}}[v]).$$

If $\overline{\mathbf{x}}[v_1] =_B^{\epsilon} 0$ or $\overline{\mathbf{x}}[v_2] =_B^{\epsilon} 0$, then $\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] \le 1/K + 2\epsilon$. From the first equation, we have $\overline{\mathbf{y}}[v] = 0$. By the second equation, we have $\overline{\mathbf{x}}[v] = 0$ and thus, $\overline{\mathbf{x}}[v] =_B^{\epsilon} 0$.

If $\overline{\mathbf{x}}[v_1] =_B^{\epsilon} 1$ and $\overline{\mathbf{x}}[v_2] =_B^{\epsilon} 1$, then $\overline{\mathbf{x}}[v_1] + \overline{\mathbf{x}}[v_2] \ge 2/K - 2\epsilon$. The first equation shows $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$. By the second equation, $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v] = 1/K \pm \epsilon$ and thus, $\overline{\mathbf{x}}[v] =_B^{\epsilon} 1$.

Proof for G_{\neg} Gates. From (1), (2) and Figure 3, we have

$$\langle \mathbf{x} | \mathbf{b}_{2k-1}^{\mathcal{S}} \rangle - \langle \mathbf{x} | \mathbf{b}_{2k}^{\mathcal{S}} \rangle = \overline{\mathbf{x}}[v_1] - \left(\overline{\mathbf{x}}_C[v_1] - \overline{\mathbf{x}}[v_1]\right), \text{ and } \langle \mathbf{a}_{2k-1}^{\mathcal{S}} | \mathbf{y} \rangle - \langle \mathbf{a}_{2k}^{\mathcal{S}} | \mathbf{y} \rangle = \left(\overline{\mathbf{y}}_C[v] - \overline{\mathbf{y}}[v]\right) - \overline{\mathbf{y}}[v].$$

If $\overline{\mathbf{x}}[v_1] = {\epsilon \atop B} 1$, then by the first equation, $\overline{\mathbf{y}}[v] = \overline{\mathbf{y}}_C[v]$. Then, by the second equation, we have $\overline{\mathbf{x}}[v] = 0$.

If $\overline{\mathbf{x}}[v_1] =_B^{\epsilon} 0$, then the first equation shows that $\overline{\mathbf{y}}[v] = 0$. By the second equation, we have $\overline{\mathbf{x}}[v] = \overline{\mathbf{x}}_C[v]$ and thus, $\overline{\mathbf{x}}[v] =_B^{\epsilon} 1$.